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K.S. Surana, D. Mysore & J.N. Reddy

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# Non-classical continuum theories for solid and fluent continua and some applications

K.S. Surana<sup>a</sup>, D. Mysore<sup>a</sup> and J.N. Reddy<sup>b</sup>

<sup>a</sup>Mechanical Engineering, University of Kansas, Lawrence, KS, USA; <sup>b</sup>Mechanical Engineering, Texas A&M University, College Station, TX, USA

## ABSTRACT

This paper presents two specific thermodynamically consistent non-classical continuum theories for solid and fluent continua. The first non-classical continuum theory for solid continua incorporates Jacobian of deformation in its entirety in the conservation and the balance laws and the derivation of the constitutive theories. The second non-classical continuum theory for solid continua considers Jacobian of deformation in its entirety as well as the Cosserat rotations in the conservation and balance laws as well as the constitutive theories. The first non-classical continuum theory for fluent continua presented here considers velocity gradient tensor in its entirety. The second non-classical continuum theory for fluent continua considers velocity gradient tensor in its entirety as well as Cosserat rotation rates in the derivation of the conservation and balance laws and the constitutive theories. Since the non-classical continuum theories for solid and fluent continua considered here incorporate additional physics of deformation due to rotations and rotation rates compared to classical continuum mechanics, the conservation and balance laws of classical continuum mechanics are shown to require modification as well as a new balance law *balance of moment of moments* is required to accommodate the new physics due to rotations and rotation rates. Eringen's micropolar, micromorphic and microstretch theories, couple stress theories and nonlocal theories are also discussed within the context of the non-classical theories presented here for solid and fluent continua. Some applications of these theories are also discussed.

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## 1 Introduction, literature review and scope of work

In classical continuum theories for a solid continua a material point has only three translational degrees of freedom. Conservation and balance law and the constitutive theories are entirely based on displacements, strain measures  $[\epsilon]$  or stretch tensor  $[S_r]$  or  $[S_l]$ . Antisymmetric part of the Jacobian of deformation  $\mathbf{J}$  or displacement gradient tensor  ${}^d\mathbf{J}$  or alternatively rotation tensor  $[R]$  in polar decomposition of the Jacobian of deformation  $[\mathbf{J}] = [\mathbf{S}_l][\mathbf{R}] = [\mathbf{R}][\mathbf{S}_r]$  are not considered in the conservation and the balance laws and the constitutive theories. Thus, classical continuum mechanics only consider partial physics out of the total physics present in  $[\mathbf{J}]$  or  $[\mathbf{J}]$ .

**CONTACT** K.S. Surana ✉ [kssurana@ku.edu](mailto:kssurana@ku.edu) Mechanical Engineering, University of Kansas, Lawrence, KS, USA

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The classical continuum theories for fluent continua consider velocities as the only three degrees of freedom at a material point or location. Conservation and the balance laws and the constitutive theories are entirely based on the velocities and symmetric part of the velocity gradient tensor. Antisymmetric part of the velocity gradient tensor is not considered in the conservation and the balance laws as well as the constitutive theories. Thus, in case of fluent continua also the classical continuum theories only incorporate the partial physics out of the total deformation physics present in the velocity gradient tensor.

Thus the currently used classical continuum theories for solid and fluent continua only consider partial physics of deformation in the conservation and the balance laws as well as the constitutive theories. There are several motivating factors for the present work. We briefly discuss the rationale, need, relevance and consistency of the work presented in this paper for solid and fluent continua. First, we consider solid continua. In any deforming solid continua Jacobian of deformation (gradients of deformed coordinates  $\bar{\mathbf{x}}$  with respect to the undeformed coordinates  $\mathbf{x}$  i.e.  $\left[\frac{\partial\{\bar{\mathbf{x}}\}}{\partial\{\mathbf{x}\}}\right] = [J]$ ) and the displacements  $\mathbf{u}$  are fundamental and complete measures of the deformation physics. In general, the Jacobian of deformation varies between material points i.e. it varies between a material point and its neighbors. Polar decomposition of the Jacobian of deformation shows that if the Jacobian of deformation varies between a material point and its neighbors so do the rotations. We could also consider decomposition of the Jacobian of deformation or displacement gradient tensor into symmetric and skew symmetric tensors. The skew symmetric tensor is a measure of pure rotation while the symmetric tensor is a measure of strain. Strain measures (such as Green's strain) are purely a function of stretch tensor or alternatively symmetric part of the Jacobian of deformation or displacement gradient tensor. In the strain measures rotations or rotation tensor plays no role. When varying rotations between the material points are resisted by the deforming solid continua conjugate moments are created. The rotation rates and the conjugate moments constitute additional mechanism of rate of work. This physics is completely absent in the currently used classical continuum theories for isotropic and homogeneous solid continua. The first non-classical continuum theory presented in this paper for solid continua incorporates  $[J]$  or  $[^dJ]$  i.e. strain measures as well as rotation measures arising due to  $[J]$  or  $[^dJ]$  in the conservation and the balance laws as well as constitutive theories. The resulting continuum theory is obviously non-classical as the classical continuum theory would have absence of rotations. The rotations in this theory arise due to  $[J]$  or  $[^dJ]$  hence are completely defined by their antisymmetric components, thus do not constitute additional degrees of freedom at a material point. The rotations are about the axis of a triad located at each material point with its axes parallel to fixed x-frame. Since these rotations naturally arise in all deforming solid continua due to  $[J]$  or  $[^dJ]$ , we refer to these as *internal rotations* as opposed to unknown additional rotational degrees of freedom (external rotations) that may be considered in some theory. Thus, this continuum theory is referred to as *non-classical continuum theory with internal rotations for solid continua*.

The second continuum theory for solid continua presented in this paper considers internal rotations as well as Cosserat rotations about the axes of the same triad considered in the non-classical theory with internal rotations. The Cosserat rotations

are additional three unknown degrees of freedom at each material point. We refer to this continuum theory as *non-classical continuum theory with internal and Cosserat rotations for solid continua*. We make several remarks in the following.

### Remarks

- (1) Internal rotations must always be considered in any continuum theory for solid continua as they always exist in all deforming solid continua.
- (2) Cosserat rotations are additional three unknown degrees of freedom at a material. The motivation for using these is to be able to describe more comprehensive physics by their inclusion in the theory as compared to a continuum theory with only internal rotations.
- (3) It is obvious that consideration of Cosserat rotations in a continuum theory without internal rotations is non-physical as internal rotations always exist at a material point.
- (4) A continuum theory that considers co-existence of internal and Cosserat rotations about the axes of a same triad at each material point must observe some restrictions. In this paper we show that in order for both of these rotations to co-exist, balance of angular momenta must incorporate moments due to both of these rotations and also the compatibility condition resulting from the second law of thermodynamics must be satisfied so that the co-existence of both rotations do not violate thermodynamic equilibrium.
- (5) The paper presents complete derivation of the conservation and the balance laws for both non-classical continuum theories, establishes need for a new balance law and presents its derivation that is applicable to both non-classical theories considered for solid continua.
- (6) We wish to point out that the deforming solid continua is isotropic and homogeneous otherwise the differential forms of the conservation and the balance laws is not possible. This significant aspect is critical to keep in mind when seeking applications of these theories.
- (7) The internal rotations and the Cosserat rotations act about the axes of the same triad, but if viewed to define different physics, then the constitutive theories for the corresponding conjugate moment tensors need to be different for the two rotations. This introduces additional material coefficients in the constitutive theories that will offer greater flexibility in terms of describing more comprehensive physics.

In the deforming fluent continua, the velocities and the velocity gradient tensor ( $\bar{\mathbf{L}}$ ) are fundamental measures of the flow physics. In general, velocity gradient tensor may vary between a location and its neighboring locations. Polar decomposition of the velocity gradient tensor into stretch rate tensor (left or right) and rotation rate tensor or alternatively its decomposition into symmetric and skew symmetric (rotation rates) shows that a varying velocity gradient tensor produces varying rotation rates between the material points. When these are resisted by the deforming fluent continua, conjugate moments are created. The rotation rates and conjugate moments constitute additional rate of work. The first continuum theory presented in this paper for fluent

continua considers velocity gradient tensor  $[\mathbf{L}]$  in its entirety in the conservation and the balance laws and the constitutive theories for fluent continua. The resulting continuum theory is obviously non-classical as the classical continuum theory would have absence of rotation rates. The rotation rates in this theory arise due to the velocity gradient tensor, hence are completely defined, thus do not constitute additional degrees of freedom at a material point. The rotation rates are about the axes of a triad located at each material point (or at each location) with its axes parallel to the fixed  $x$ -frame. Since these rotation rates naturally arise in all deforming fluent continua, we refer to these as *internal rotation rates* as opposed to unknown rotational rates as additional degrees of freedom (external rotation rates) that may be considered in some theory. Thus, this continuum theory is referred to as *non-classical continuum theory with internal rotation rates for fluent continua*.

The second continuum theory for fluent continua presented in this paper considers internal rotation rates as well as Cosserat rotation rates about the axes of the same triad considered in the internal rotation rate theory. The Cosserat rotation rates are additional three unknown degrees of freedom at a material point (or at a location). We refer to this continuum theory as *non-classical continuum theory with internal and Cosserat rotation rates for fluent continua*. We make several remarks in the following. These are parallel to the remarks made for solid continua, but stated here for completeness.

### Remarks

- (1) Internal rotation rates must always be considered in any continuum theory for fluent continua as they always exist in all deforming fluent continua.
- (2) Cosserat rotation rates are additional three unknown degrees of freedom at a material. The motivation for considering these is to be able to describe more comprehensive physics by their inclusion in the theory as compared to a continuum theory with only internal rotation rates.
- (3) It is obvious that consideration of Cosserat rotation rates without internal rotation rates is non-physical as the internal rotation rates are always present in a deforming fluent continua.
- (4) A continuum theory that considers co-existence of internal and Cosserat rotation rates about the axes of a same triad must observe some restrictions. In this paper we show that in order for both of these rotation rates to co-exist, balance of angular momenta must incorporate moments due to both of these rotation rates and also the compatibility condition resulting from the second law of thermodynamics must be satisfied so that the co-existence of both rotation rates do not violate thermodynamic equilibrium.
- (5) Complete derivations of the conservation and the balance laws as well as constitutive theories are presented for both non-classical continuum theories. Need for additional balance law due to introduction of new physics is established and the derivation of new balance law, *balance of moment of moments* is presented.
- (6) As in case of solid continua, here also the deforming fluent continua is homogeneous and isotropic otherwise the differential forms of the conservation and

the balance laws is not possible. This aspect of the theories is important to keep in mind when seeking their applications.

- (7) The internal rotation rates and the Cosserat rotation rates act about the axes of the same triad, but if viewed to define different physics, then the constitutive theories for the corresponding conjugate moment tensors need to be different for the two rotation rates. This introduces additional material coefficients in the constitutive theories for the moment tensors that will offer greater flexibility in terms of describing more comprehensive physics.

In the following we present a brief literature review of published works on non-classical continuum theories under the titles micropolar theories, nonlocal theories, couple stress theories, etc. We discuss relevance and correspondence of these theories to the present work after we present the literature review. A comprehensive treatment of micropolar theories can be found in references [1–21] that are designed to accommodate effects at scales smaller than continuum scale. Micropolar theories consider measures of microdeformation due to microconstituents in the continuum. The concept of couple stresses is presented by Koiter [17]. Balance laws for micromorphic materials are presented in reference [18]. The micropolar theories consider micro deformation due to microconstituents in the continuum. In references [22–24] by Reddy et. al. and reference [25] by Zang et. al. nonlocal theories are presented for bending, buckling and vibration of beams, beams with nanocarbon tubes and bending of plates. The nonlocal effects are believed to be incorporated due to the work presented by Eringen [6] in which definition of a nonlocal stress tensor is introduced through integral relationship using the product of macroscopic stress tensor and a distance kernel representing the nonlocal effects. The concept of couple stresses was introduced by Voigt in 1881 by assuming existence of a couple or moment per unit area on the oblique plane of the deformed tetrahedron in addition to the stress or force per unit area. Since the introduction of this concept many published works have appeared. We cite some recent works, most of which are related to micropolar couple stress theories. Authors in reference [26] report experimental study of micropolar and couple stress elasticity of compact bones in bending. Conservation integrals in couple stress elasticity are reported in reference [27]. A microstructure-dependent Timoshenko beam model based on modified couple stress theories is reported by Ma et. al [28]. Further account of couple stress theories in conjunction with beams can be found in references [29–31]. Treatment of rotation gradient dependent strain energy and its specialization to Von Kármán plates and beams can be found in reference [32]. Other accounts of micropolar elasticity and Cosserat modeling of cellular solids can be found in references [33–35]. We remark that in references [26–35], Lagrangian description is used for solid matter, however the mathematical descriptions are purely derived using strain energy density functional and principle of virtual work. This approach works well for elastic solids in which mechanical deformation is reversible. Extension of these works to thermoviscoelastic solids with and without memory is not possible. In such materials the thermal field and mechanical deformation are coupled due to the fact that the rate of work results in rate of entropy production. In reference [36] Altenbach and Eremeyev present a linear theory for micropolar plates. Each material point is regarded as a small rigid body with six degrees of freedom. Kinematics of plates is described using the vector of translations

and the vector of rotations as dependent variables. Equations of equilibrium are established in  $\mathbb{R}^3$  and  $\mathbb{R}^2$ . Strain energy density function is used to present linear constitutive theory. The mathematical models of reference [37] are extended by the same authors to present strain rate tensors and the constitutive equations for inelastic micropolar materials. In reference [38], authors consider the conditions for the existence of the acceleration waves in thermoelastic micropolar media. The work concludes that the presence of the energy equation with Fourier heat conduction law does not influence the wave physics in thermoelastic micropolar media. Thus, from the point of view of acceleration waves in thermoelastic polar media, thermal effects i.e. temperature can be treated as a parameter. In reference [39], authors present a collection of papers related to the mechanics of continua dealing with micro-macro aspects of the physics (largely related to solid matter). In reference [40] a micropolar theory is presented for binary media with applications to phase-transitional flow of fiber suspensions. Such flows take place during the filling state of injection molding of short fiber reinforced thermoplastics. A similarity solution for boundary layer flow of a polar fluid is given in reference [41]. In specific the paper borrows constitutive equations that are claimed to be valid for flow behavior of a suspension of very fine particles in a viscous fluid. Kinematics of micropolar continuum is presented in reference [42]. References [43,44] consider material symmetry groups for linear Cosserat continuum and non-linear polar elastic continuum. Grekova et. al [45–47]. consider various aspects of wave processes in ferromagnetic medium and elastic medium with micro-rotations as well as some aspects of linear reduced Cosserat medium. In references [48–66] various aspects of the kinematics of micropolar theories, couple stress theories, etc. are discussed and presented including some applications to plates and shells.

There is much more similar published works regarding the polar, couple stress and nonlocal theories utilizing concepts similar to those already discussed here. We make some remarks in the following.

### ***1.1. Non-classical continuum theory based on internal rotations and couple stress theories for solid continua***

In a deforming solid matter displacement gradient tensor ( ${}^d\mathbf{J}$ ) is a complete measure of deformation physics, hence must form the foundation of conservation and balance laws. Decomposition of  ${}^d\mathbf{J}$  into symmetric tensor (measure of strains) and antisymmetric tensor (measure of internal rotations) suggests that incorporating  ${}^d\mathbf{J}$  in its entirety in the conservation and the balance laws implies that we must incorporate internal rotations in addition to the strain measures in deriving the conservation and the balance laws. The internal rotations vary from a material point to the neighboring material points as  ${}^d\mathbf{J}$  varies. When these varying internal rotations are resisted by the deforming matter internal moments are created. This physics exists in all deforming solid matter, in some to lesser degree than others, but is always present, otherwise in the absence of the moment tensor the material points will simply experience rigid rotations with respect to their neighbors. The continuum theory that accounts for this deformation physics of internal rotations and associated moments is referred to as non-classical continuum theory with internal rotations.

If we consider a bounded volume of matter disturbed by force or forces on its boundary (and not by the moments that are independent of the forces), then the internal rotations and associated moments are created in the deformed volume. These are obviously equilibrating within the deformed volume without the existence of moment on the boundary of the volume. If one consider a tetrahedron with its oblique plane representing external surface of the volume and its internal planes parallel to the  $x$ -frame, then Cauchy principle establishes relationship between the moment on the oblique plane and those on the planes of the tetrahedron (Cauchy moment tensor). If there is no moment on the oblique plane of the tetrahedron, then the component of the Cauchy moment tensor are not zero but are self equilibrating. This is the fundamental difference between the couple stress theories in which one assumes existence of a moment tensor (couple) on the oblique plane of the tetrahedron that is the cause of the non-symmetric stress tensor (couple stress tensor), and Cauchy moment tensor. In such theories if there is no moment on the oblique plane of the tetrahedron, then the Cauchy moment tensor is zero and the Cauchy stress tensor is symmetric and we have standard classical continuum theory. This is obviously not the case in the non-classical continuum theory based on internal rotations that are always present regardless of the moment on the boundary surface of the deforming volume. Hence, the conservation and the balance laws in the present work for non-classical continuum theory with internal rotations are unaffected whether there is a moment acting on the boundary of the deforming volume. This is obviously not the case in couple stress theories in which existence of the moment on the boundary of the volume is a necessary requirement.

### ***1.2. Non-classical continuum theory based on internal rotation rates and the couple stress theories for fluent continua***

In a deforming volume of fluid, velocity gradient tensor  $\bar{\mathbf{L}}$  is a complete measure of deformation physics, hence must form the foundation of the conservation and the balance laws. Decomposition of  $\bar{\mathbf{L}}$  into symmetric tensor (measures of strain rates) and antisymmetric tensor (measures of internal rotation rates) suggests that incorporating  $\bar{\mathbf{L}}$  in its entirety in the conservation and the balance laws implies that we must incorporate internal rotation rates in addition to the strain rate measures in deriving the conservation and the balance laws. The internal rotation rates vary from a material point to the neighboring material points due to the fact that  $\bar{\mathbf{L}}$  varies. When these varying material rotation rates are resisted by the deforming matter, internal moments are created. This physics exists in all deforming fluent continua to some degree. The continuum theory for fluent continua that accounts for this deformation physics of internal rotation rates and associated moments is referred to as non-classical continuum theory with internal rotation rates for fluent continua.

If we consider a bounded volume of matter disturbed by a force or forces on its boundary (and not by the moments that are independent of the forces), then the internal rotation rates and the moments are created in the deformed volume. These are obviously equilibrating within the deformed volume without the existence of the moment on the boundary of the volume. If one considers a tetrahedron with its oblique plane representing external surface of



the volume and its internal planes parallel to the  $x$ -frame, then the Cauchy principle establishes a relationship between the moment on the oblique plane of the tetrahedron and those on the planes of the tetrahedron (Cauchy moment tensor). If there is no moment on the oblique plane of the tetrahedron, the components of the Cauchy moment tensor are not zero but are self equilibrating. This is the fundamental difference between the couple stress theories in which one assumes existence of the moment (couple) on the oblique plane of the tetrahedron that is the cause of the non-symmetric stress tensor (couple stress tensor), and Cauchy moment tensor. In such theories if there is no moment on the oblique plane of the tetrahedron, then the Cauchy moment tensor is zero and the Cauchy stress tensor is symmetric and we have standard classical continuum theory for fluent continua. This is obviously not the case in the non-classical continuum theory based on internal rotation rates that are always present regardless of the moment on the bounding surface of the deforming volume. Hence, the conservation and the balance laws in the present work for non-classical continuum theory with internal rotation rates are unaffected whether there is a moment acting on the boundary of the deforming volume. This is obviously not the case in couple stress theories for fluent continua in which existence of the moment on the boundary of the volume is a necessary requirement.

## 2 Notations, measures of rotations, rotation rates and their gradients

In the following we present details of the measures of rotations, rotation rates and their gradients for solid and fluent continua. Both internal and Cosserat rotations, their rates and gradients are considered.

### 2.1 Solid continua

Following reference [67] quantities with an over-bar are quantities in the current (deformed) configuration (i.e., all quantities with over-bar are functions of coordinates  $\bar{x}_i$  and time  $t$ , the Eulerian description). Quantities without an over-bar are quantities referred to the reference configuration (i.e., these are functions of undeformed coordinates  $x_i$  and time  $t$ , Lagrangian description). The configuration at time  $t = t_0 = 0$ , commencement of evolution, is considered as the reference configuration. Thus,  $x_i$  and  $\bar{x}_i$  are coordinates of the same material point in reference and current configurations, respectively, both measured in a fixed Cartesian  $x$ -frame. For solid continua we only consider the Lagrangian description.

Consider the Jacobian of deformation defined by  $\mathbf{J} = \mathbf{e}_i \otimes \mathbf{e}_j \frac{\partial \bar{x}_j}{\partial x_i}$ . The rows are the covariant base vectors, whereas in Murnaghan's notation  $[J] = \left[ \frac{\partial \{\bar{x}\}}{\partial \{x\}} \right] = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ x_1 & x_2 & x_3 \end{bmatrix}$ , the columns are the covariant base vectors (i.e., in this definition  $[J]$  is the transpose of  $\mathbf{J}$  in the first definition). Both definitions are obviously covariant measures in the Lagrangian description. Likewise,  $\bar{\mathbf{J}} = \mathbf{e}_i \otimes \mathbf{e}_j \frac{\partial x_j}{\partial \bar{x}_i}$  and  $[\bar{J}] = \left[ \frac{\partial \{x\}}{\partial \{\bar{x}\}} \right] = \begin{bmatrix} x_1 & x_2 & x_3 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \end{bmatrix}$  are also Jacobians of deformation but they are contravariant measures in the Eulerian description. Columns of  $\bar{\mathbf{J}}$  are the contravariant base vectors whereas in case of  $[\bar{J}]$  its rows are the contravariant base vectors (i.e.,  $\bar{\mathbf{J}}$  is transpose of  $[\bar{J}]$ ). In this paper the Jacobians denoted by  $[J]$  and  $[\bar{J}]$  are used.

Since the work presented in this paper only considers small strain and small deformations, the distinction between covariant and contravariant measures disappears as  $\bar{x}_i \simeq x_i$  (i.e., the deformed configuration is not substantially different from the undeformed configuration). For both deformation measured,  $\det[J] = \det[\bar{J}] \cong 1$  and, hence, in the development of the theory there is a need to separate displacements from the deformed coordinates. The displacement gradient  $[^d J]$  is defined as

$$[^d J] = \frac{\partial \{u\}}{\partial \{x\}} = \begin{bmatrix} u_1 & u_2 & u_3 \\ x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \quad (1)$$

The Cauchy stress tensor is used as a measure of stress because the deformed and undeformed tetrahedron can be treated the same for small deformation. Hence, the conservation and balance laws must be based entirely of  $[^d J]$  (i.e.  $[^d_s J]$  and  $[^d_a J]$  both must be considered in the conservation and balance laws).

The displacement gradient  $[^d J]$  can be written in component form as

$$^d J_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) + \frac{1}{2} (u_{i,j} - u_{j,i}) = ^d_s J_{ij} + ^d_a J_{ij} \quad (2)$$

in which

$$[^d_a J] = \begin{bmatrix} 0 & {}_i\Theta_{x3} & -{}_i\Theta_{x2} \\ -{}_i\Theta_{x3} & 0 & {}_i\Theta_{x1} \\ {}_i\Theta_{x2} & -{}_i\Theta_{x1} & 0 \end{bmatrix} \quad (3)$$

$${}_i\Theta_{x1} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right); \quad {}_i\Theta_{x2} = \frac{1}{2} \left( \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \right); \quad {}_i\Theta_{x3} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \quad (4)$$

rotations  ${}_i\Theta_{x1}$ ,  ${}_i\Theta_{x2}$ ,  ${}_i\Theta_{x3}$  are referred to as internal rotations.

Alternatively (4) can be derived as

$$\nabla \times \mathbf{u} = \mathbf{e}_i \times \mathbf{e}_j \frac{\partial u_j}{\partial x_i} = \epsilon_{ijk} \mathbf{e}_k \frac{\partial u_j}{\partial x_i} \quad (5)$$

$$\nabla \times \mathbf{u} = \mathbf{e}_1 \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) + \mathbf{e}_2 \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) + \mathbf{e}_3 \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \quad (6)$$

$$\nabla \times \mathbf{u} = \mathbf{e}_1 (-2{}_i\Theta_{x1}) + \mathbf{e}_2 (-2{}_i\Theta_{x2}) + \mathbf{e}_3 (-2{}_i\Theta_{x3}) \quad (7)$$

The sign difference in (4) and (7) is due to the fact that rotations in (4) are in clockwise sense, whereas quantities in (6) are twice the magnitude compared to those in (4) and are in counterclockwise sense. In this paper, (4) is considered as the definition of rotations (i.e., clockwise). The rotations defined in (4) exist at every material point in the deforming solid.

On the other hand using polar decomposition we can obtain the right and the left stretch tensors  $[^d S_r]$  and  $[^d S_l]$  that are symmetric and positive-definite, and  $[^d R]$  that is an orthogonal rotation tensor, a rotation matrix corresponding to the rotation angles defined in (4).  $[^d J]$  and  $[^d R]$  contain the same physics as these both are derived from

$[^dJ]$  but in different forms.  $[^d_aJ]$  contains rotation angles while  $[^dR]$  is the corresponding rotation matrix or tensor. Both in their forms given here can be used in derivations as needed. The same holds true for  $[R]$  and  $[_aJ]$  derived from  $[J]$ . However, deriving  $[^dR]$  from  $[^d_aJ]$  (or  $[R]$  from  $[_aJ]$ ) or vice versa in general in  $\mathbb{R}^3$  may not be possible or unique. Fortunately there is no need for this here.

Incorporating  $[^dJ]$  in its entirety in the derivation of conservation and balance laws implies incorporating  $[^d_sJ]$ , and  $[^d_aJ]$  (i.e., rotations  ${}_i\Theta_{x1}$ ,  ${}_i\Theta_{x2}$ , and  ${}_i\Theta_{x3}$  about the axes of a triad located at each material point) both. Rotations in  $[^d_aJ]$  are internal and are completely defined by skew-symmetric part of  $[^dJ]$ .

Let  ${}_e\Theta_{x1}$ ,  ${}_e\Theta_{x2}$ , and  ${}_e\Theta_{x3}$  be the additional Cosserat rotations (unknown) about the same axes of the triad as used for internal rotations  ${}_i\Theta$ , assumed positive counterclockwise. Let  $[^e_aY]$  be the antisymmetric matrix of rotation angles defined using rotations  ${}_e\Theta_i$ , then

$$[^e_aY] = \begin{bmatrix} 0 & {}_e\Theta_{x3} & -{}_e\Theta_{x2} \\ -{}_e\Theta_{x3} & 0 & {}_e\Theta_{x1} \\ {}_e\Theta_{x2} & -{}_e\Theta_{x1} & 0 \end{bmatrix} \quad (8)$$

Angles  ${}_e\Theta_j$  in (8) are positive when counterclockwise. Let

$$[\mathbb{J}] = [^d_sJ] + [^d_aJ] - [^e_aY] \quad (9)$$

$$[\mathbb{J}] = [^d_sJ] + [^t_a r] \quad (10)$$

$$[^t_a r] = [^d_aJ] - [^e_aY] = \begin{bmatrix} 0 & {}^t\Theta_{x3} & -{}^t\Theta_{x2} \\ -{}^t\Theta_{x3} & 0 & {}^t\Theta_{x1} \\ {}^t\Theta_{x2} & -{}^t\Theta_{x1} & 0 \end{bmatrix} \quad (11)$$

in which  $[^t_a r]$  is the antisymmetric matrix containing total rotations  ${}^t\Theta_{x1}$ ,  ${}^t\Theta_{x2}$ , and  ${}^t\Theta_{x3}$  about the axes of the triad at a material point, considered positive in the clockwise sense. Obviously,

$$\begin{aligned} {}^t\Theta_{x1} &= {}_i\Theta_{x1} - {}_e\Theta_{x1} \\ {}^t\Theta_{x2} &= {}_i\Theta_{x2} - {}_e\Theta_{x2} \\ {}^t\Theta_{x3} &= {}_i\Theta_{x3} - {}_e\Theta_{x3} \end{aligned} \quad (12)$$

Due to varying  $[\mathbb{J}]$  between material points, total rotations  ${}^t\Theta$  vary between neighboring material points. When these are resisted by the deforming matter, conjugate moments are generated which, together with  ${}^t\Theta$  and their rates, result in additional energy storage and/or dissipation as well as additional rheology.

### Remarks

- (1)  $[^d_sJ]$  represents the usual infinitesimal strain tensor as in the linear theory of elasticity.
- (2)  $[^d_aJ]$ ,  $[^e_aY]$ , and  $[^t_a r]$  are antisymmetric tensors containing rotation angles, hence are not measures of strain.
- (3) Based on (1) and (2),  $[\mathbb{J}]$  is not a strain tensor, but rather addition of strain tensor  $[^d_sJ]$  and the internal and Cosserat rotation angle tensors  $[^d_aJ]$  and  $[^e_aY]$ .

If we let

$$\{ {}_t\Theta \}^T = [ {}_t\Theta_{x1}, {}_t\Theta_{x2}, {}_t\Theta_{x3} ] \quad (13)$$

Gradients of total rotations in (13) ( $[{}^\Theta J^t]$ ) can be defined using

$$[{}^\Theta J^t] = \left[ \frac{\partial \{ {}_t\Theta \}}{\partial \{ x \}} \right] \quad \text{or} \quad {}^\Theta J_{ij}^t = \frac{\partial {}_t\Theta_i}{\partial x_j} \quad (14)$$

The gradient tensor  $[{}^\Theta J^t]$  of total rotations can be decomposed into symmetric and antisymmetric parts  $[{}_s{}^\Theta J^t]$  and  $[{}_a{}^\Theta J^t]$ .

$$[{}^\Theta J^t] = [{}_s{}^\Theta J^t] + [{}_a{}^\Theta J^t] \quad (15)$$

$$\begin{aligned} [{}_s{}^\Theta J^t] &= \frac{1}{2} \left( [{}^\Theta J^t] + [{}^\Theta J^t]^T \right) \\ [{}_a{}^\Theta J^t] &= \frac{1}{2} \left( [{}^\Theta J^t] - [{}^\Theta J^t]^T \right) \end{aligned} \quad (16)$$

It is well known that the rotation gradient tensor is a tensor of rank three, but a vector representation of rotations in (13) is helpful as it results in the rotation gradient tensor of rank two.

## 2.2 Fluent continua

The conservation and balance laws for fluent continua are considered in Eulerian description. Velocities ( $\bar{\mathbf{v}}$ ) and the velocity gradient tensor  $\left[ \frac{\partial \{ \bar{\mathbf{v}} \}}{\partial \{ \bar{\mathbf{x}} \}} \right] = [\bar{\mathbf{L}}]$  are fundamental measures of deformation physics in fluent continua. Decomposition of  $[\bar{\mathbf{L}}]$  into symmetric ( $[\bar{\mathbf{D}}]$ ) and antisymmetric ( $[\bar{\mathbf{W}}]$ ) tensors gives

$$[\bar{\mathbf{L}}] = \left[ \frac{\partial \{ \bar{\mathbf{v}} \}}{\partial \{ \bar{\mathbf{x}} \}} \right] = [\bar{\mathbf{D}}] + [\bar{\mathbf{W}}] \quad (17)$$

$$[\bar{\mathbf{D}}] = \frac{1}{2} \left( [\bar{\mathbf{L}}] + [\bar{\mathbf{L}}]^T \right); \quad [\bar{\mathbf{W}}] = \frac{1}{2} \left( [\bar{\mathbf{L}}] - [\bar{\mathbf{L}}]^T \right) \quad (18)$$

or

$$\bar{D}_{ij} = \frac{1}{2} (\bar{v}_{i,j} + \bar{v}_{j,i}); \quad \bar{W}_{ij} = \frac{1}{2} (\bar{v}_{i,j} - \bar{v}_{j,i}) \quad (19)$$

Expanded form of  $[\bar{\mathbf{W}}]$  can be written as

$$[\bar{\mathbf{W}}] = \begin{bmatrix} 0 & {}^t\bar{\Theta}_{x3} & -{}_i\bar{\Theta}_{x2} \\ -{}_i\bar{\Theta}_{x3} & 0 & {}^t\bar{\Theta}_{x1} \\ {}^t\bar{\Theta}_{x2} & -{}_i\bar{\Theta}_{x1} & 0 \end{bmatrix} \quad (20)$$

$${}_i\bar{\Theta}_{x1} = \frac{1}{2} \left( \frac{\partial \bar{v}_2}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_2} \right); \quad {}^t\bar{\Theta}_{x2} = \frac{1}{2} \left( \frac{\partial \bar{v}_3}{\partial \bar{x}_1} - \frac{\partial \bar{v}_1}{\partial \bar{x}_3} \right); \quad {}^t\bar{\Theta}_{x3} = \frac{1}{2} \left( \frac{\partial \bar{v}_1}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_1} \right) \quad (21)$$

Alternatively, notations in (21) can be derived as

$$\bar{\nabla} \times \bar{\mathbf{v}} = \mathbf{e}_i \times \mathbf{e}_j \frac{\partial \bar{v}_j}{\partial \bar{x}_i} = \epsilon_{ijk} \mathbf{e}_k \frac{\partial \bar{v}_j}{\partial \bar{x}_i} \quad (22)$$

$$\bar{\nabla} \times \bar{\mathbf{v}} = \mathbf{e}_1 \left( \frac{\partial \bar{v}_3}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_3} \right) + \mathbf{e}_2 \left( \frac{\partial \bar{v}_1}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_1} \right) + \mathbf{e}_3 \left( \frac{\partial \bar{v}_2}{\partial \bar{x}_1} - \frac{\partial \bar{v}_1}{\partial \bar{x}_2} \right) \quad (23)$$

or

$$\bar{\nabla} \times \bar{\mathbf{v}} = \mathbf{e}_1 (-2({}^t\bar{\Theta}_{x1})) + \mathbf{e}_2 (-2({}^t\bar{\Theta}_{x2})) + \mathbf{e}_3 (-2({}^t\bar{\Theta}_{x3})) \quad (24)$$

The rotation rates in (21) are in clockwise sense, whereas quantities in (24) are twice the magnitude compared to (21) and are in counterclockwise sense. We note that  $[\bar{W}]$ , the antisymmetric part of  $[\bar{L}]$ , has rotation rates whereas  $[{}^t\bar{R}]$  from the polar decomposition of  $[\bar{L}]$  is a transformation matrix related to rotation rates. The details in both are related to rotation rates and are derived using  $[\bar{L}]$ , hence use of  $[{}^t\bar{R}]$  or  $[\bar{W}]$  is interchangeable depending upon the need. Another important point we note is that from (20),  $[\bar{W}]$  is undoubtedly a tensor of rank two. This is also obvious from (22) containing  $\mathbf{e}_i \times \mathbf{e}_j$  term. However, the rotation rates  ${}^t\bar{\Theta}_i$  as in (21) can be viewed as a vector quantity. That is, the three rotations about the axes of a triad at a material point can be arranged in the form of a vector. This form is advantageous when determining gradients of the rotation rates (shown later). We clearly observe that  ${}^t\bar{\Theta}_i$  are completely defined by the components of  $[\bar{L}]$ , i.e. dependent on the components of  $[\bar{L}]$ , therefore are not unknown degrees of freedom at a material point or at a location.

Let  ${}^t_e\bar{\Theta}_{x1}$ ,  ${}^t_e\bar{\Theta}_{x2}$ ,  ${}^t_e\bar{\Theta}_{x3}$  (or  ${}^t_e\bar{\Theta}$ ) be the additional Cosserat rotation rates (unknown) assumed positive when counterclockwise about the axes of the same triad as used for internal rotation rates  ${}^t\bar{\Theta}$ . Let  $[_e\bar{W}]$  be the antisymmetric tensor of rotation rates defined using  ${}^t_e\bar{\Theta}$  (similar to  $[\bar{W}]$  in (20) defined using  ${}^t\bar{\Theta}$ ).

$$[_e\bar{W}] = \begin{bmatrix} 0 & {}^t_e\bar{\Theta}_{x3} & -{}^t_e\bar{\Theta}_{x2} \\ -{}^t_e\bar{\Theta}_{x3} & 0 & {}^t_e\bar{\Theta}_{x1} \\ {}^t_e\bar{\Theta}_{x2} & -{}^t_e\bar{\Theta}_{x1} & 0 \end{bmatrix} \quad (25)$$

Angles  ${}^t_e\bar{\Theta}$  in (25) are positive when counterclockwise. Let

$$[\bar{\mathbb{L}}] = [\bar{D}] + [\bar{W}] - [_e\bar{W}] = [\bar{D}] + [{}_t\bar{W}] \quad (26)$$

$$[\bar{\mathbb{L}}] = [\bar{L}] - [_e\bar{W}] \quad (27)$$

hence

$$[\bar{L}] = [\bar{\mathbb{L}}] + [_e\bar{W}] \quad (28)$$

where

$$[{}_t\bar{W}] = [\bar{W}] - [_e\bar{W}] = \begin{bmatrix} 0 & {}^t\bar{\Theta}_{x3} & -{}^t\bar{\Theta}_{x2} \\ -{}^t\bar{\Theta}_{x3} & 0 & {}^t\bar{\Theta}_{x1} \\ {}^t\bar{\Theta}_{x2} & -{}^t\bar{\Theta}_{x1} & 0 \end{bmatrix} \quad (29)$$

$[_t\bar{W}]$  is the antisymmetric tensor containing total rotation rates  ${}^t\bar{\Theta}_{x1}$ ,  ${}^t\bar{\Theta}_{x2}$ ,  ${}^t\bar{\Theta}_{x3}$  about the axes of a triad at a material point (or a location), considered positive in the clockwise sense. Obviously

$$\begin{aligned} {}^t\bar{\Theta}_{x1} &= {}^t\bar{\Theta}_{x1} - {}^e\bar{\Theta}_{x1} \\ {}^t\bar{\Theta}_{x2} &= {}^t\bar{\Theta}_{x2} - {}^e\bar{\Theta}_{x2} \\ {}^t\bar{\Theta}_{x3} &= {}^t\bar{\Theta}_{x3} - {}^e\bar{\Theta}_{x3} \end{aligned} \quad (30)$$

Due to varying  $[_t\bar{W}]$  between neighboring material points (accounting for internal rotation rates and Cosserat rotation rates) or neighboring locations, total rotation rates  ${}^t\bar{\Theta}$  vary between them also. When these are resisted by the deforming matter, conjugate moment tensor is generated which, together with  ${}^t\bar{\Theta}$  may result in additional dissipation and/or rheology (memory).

### Remarks

- (1)  $[\bar{D}]$  represents the usual strain rate tensor (symmetric part of the velocity gradient tensor) used in fluid mechanics.
- (2)  $[\bar{W}]$ ,  $[_e\bar{W}]$ ,  $[_t\bar{W}]$  are antisymmetric tensor containing rotation rates, hence are not measures of strain rate.
- (3) Based on (1) and (2)  $[\bar{L}]$  is not a strain rate tensor, but rather addition of strain rate tensor  $[\bar{D}]$  and the internal and Cosserat rotation rates  $[\bar{W}]$  and  $[_e\bar{W}]$ .

The rotation rate tensors are tensors of rank two. This is obvious from the definitions of  $[\bar{W}]$ ,  $[_e\bar{W}]$ ,  $[_t\bar{W}]$ . For example, definition of  $\bar{W}$  from (18) or (19) clearly shows that it is a tensor of rank two, i.e.

$$\bar{W} = \frac{1}{2} \mathbf{e}_i \otimes \mathbf{e}_j \left( \frac{\partial \bar{v}_j}{\partial \bar{x}_i} - \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \right) \quad (31)$$

The gradient of  $\bar{W}$  in (31) can be written as

$$\bar{\nabla} \bar{W} = \mathbf{e}_l \frac{\partial}{\partial \bar{x}_l} \otimes \left( \frac{1}{2} \mathbf{e}_i \otimes \mathbf{e}_j \left( \frac{\partial \bar{v}_j}{\partial \bar{x}_i} - \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \right) \right) = \frac{1}{2} \mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j \frac{\partial}{\partial \bar{x}_l} \left( \frac{\partial \bar{v}_j}{\partial \bar{x}_i} - \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \right) \quad (32)$$

Clearly  $\bar{\nabla} \bar{W}$ , i.e. gradient of  $\bar{W}$ , is a tensor of rank three. Likewise, gradients of  ${}^t\bar{\Theta}$  and  ${}^e\bar{\Theta}$  are also tensors of rank three. An alternative presentation of the gradients of  ${}^t\bar{\Theta}$  is simple and easier to incorporate in the further developments. Let us represent total rotation rates as a vector

$$\{ {}^t\bar{\Theta} \}^T = [ {}^t\bar{\Theta}_{x1}, {}^t\bar{\Theta}_{x2}, {}^t\bar{\Theta}_{x3} ] \quad (33)$$

Gradients of  ${}^t\bar{\Theta}$  in (33) can be defined using

$$[ {}^t\bar{\Theta} ] = \frac{\partial \{ {}^t\bar{\Theta} \}}{\partial \{ \bar{x} \}} \quad \text{or} \quad {}^t\bar{\Theta}_{ij} = \frac{\partial ({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} \quad (34)$$

The gradient tensor  ${}^t\bar{\Theta}J$  of total rotation rates in (34) can be decomposed into symmetric and antisymmetric tensors  $[ {}^t\bar{\Theta} ]_s$  and  $[ {}^t\bar{\Theta} ]_a$ .

$$[{}^t\bar{\theta}\bar{J}] = [{}^t\bar{\theta}J] + [{}^t\bar{\theta}\bar{J}] \quad (35)$$

$$\begin{aligned} [{}^t\bar{\theta}J] &= \frac{1}{2} \left( [{}^t\bar{\theta}J] + [{}^t\bar{\theta}J]^T \right) \\ [{}^t\bar{\theta}\bar{J}] &= \frac{1}{2} \left( [{}^t\bar{\theta}J] - [{}^t\bar{\theta}J]^T \right) \end{aligned} \quad (36)$$

### 3 Considerations of stress and moment tensors

#### 3.1 Solid continua

When the gradients of displacements vary between neighboring material points, so do the internal rotations  ${}^dJ$  and likewise the Cosserat rotations  ${}^e\mathbf{y}$  may also vary between the neighboring material points. Hence, total rotation tensor  ${}^t\mathbf{r}$  can vary between the material points. When rotations  ${}^t\mathbf{r}$  are resisted by the deforming matter conjugate moments are created.  ${}^t\mathbf{r}$  and their rates and conjugate moments can result in additional energy storage, dissipation, and rheology, i.e. in addition to those which are already present due to Cauchy stress tensor, strain, and strain rate tensors. Thus in the deforming matter total rotations  ${}^t\mathbf{r}$  are conjugate to moment tensor which necessitates that on the boundary of the deformed volume there must exist a resultant moment.

Consider a volume of matter  $V$  in the reference configuration with closed boundary  $\partial V$ . The volume  $V$  is isolated from  $\tilde{V}$  by a hypothetical surface  $\partial V$  as in the cut principle of Cauchy. Consider a tetrahedron  $T_1$  such that its oblique plane is part of  $\partial V$  and its other three planes are orthogonal to each other and parallel to the planes of the  $x$ -frame. Upon deformation,  $V$  and  $\partial V$  occupy  $\bar{V}$  and  $\partial \bar{V}$  and likewise  $V$  and  $\partial V$  deform into  $\bar{V}$  and  $\partial \bar{V}$ . The tetrahedron  $T_1$  deforms into  $\bar{T}_1$  whose edges (under finite deformation) are non-orthogonal covariant base vectors  $\tilde{g}_i$ . The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point  $\bar{o}$  so that the assumption of the oblique plane  $\bar{A}\bar{B}\bar{C}$  being flat but still part of  $\partial \bar{V}$  is valid. When the deformed tetrahedron is isolated from volume  $\bar{V}$  it must be in equilibrium under the action of disturbance on surface  $\bar{A}\bar{B}\bar{C}$  from the volume surrounding  $\bar{V}$  and the internal fields that act on the flat faces which equilibrate with the mating faces in volume  $\bar{V}$  when the tetrahedron  $T_2$  is placed back in the volume  $\bar{V}$ .

Consider the deformed tetrahedron  $\bar{T}_1$ . Let  $\bar{\mathbf{P}}$  be the average stress per unit area on plane  $\bar{A}\bar{B}\bar{C}$ ,  $\bar{\mathbf{M}}$  be the average moment per unit area on plane  $\bar{A}\bar{B}\bar{C}$  henceforth referred to as moment for short, and  $\bar{\mathbf{n}}$  be the unit normal to the face  $\bar{A}\bar{B}\bar{C}$ .  $\bar{\mathbf{P}}$ ,  $\bar{\mathbf{M}}$ , and  $\bar{\mathbf{n}}$  all have different directions when the deformation is finite. Based on the small deformation assumption, the deformed coordinates  $\bar{x}_i$  are approximately same as undeformed coordinates  $x_i$ , thus the deformed tetrahedron  $\bar{T}_1$  in the current configuration is close to its map  $T_1$  in the reference configuration. With this assumption all stress measures (first and second Piola-Kirchhoff stress tensors, Cauchy stress tensor) are approximately the same. The same holds for the moment tensors. Thus with the assumption  $\bar{\mathbf{x}} \simeq \mathbf{x}$  we can write

$$\bar{\mathbf{P}} = \mathbf{P}, \quad \bar{\mathbf{M}} = \mathbf{M} \quad (37)$$

The Cauchy principle for  $\bar{\mathbf{P}}$  and  $\bar{\mathbf{M}}$  gives (hence for  $\mathbf{P}$  and  $\mathbf{M}$ )

$$\mathbf{P} = \boldsymbol{\sigma}^T \cdot \mathbf{n}, \quad \mathbf{M} = \mathbf{m}^T \cdot \mathbf{n} \quad (38)$$

in which  $\boldsymbol{\sigma}$  is Cauchy stress tensor and  $\mathbf{m}$  is Cauchy moment tensor (per unit area).

### 3.2 Fluent continua

As well known, the Cauchy stress measures in the Eulerian description using the deformed tetrahedron can be contravariant or covariant [67]. We describe these here.

Consider the deformed tetrahedron  $\bar{T}_1$ . Let  $\bar{\mathbf{P}}$  be the average stress per unit area on plane  $\bar{ABC}$ ,  $\bar{\mathbf{M}}$  be the average moment per unit area on plane  $\bar{ABC}$  (henceforth referred to as moment for short), and  $\bar{\mathbf{n}}$  be the unit exterior normal to the face  $\bar{ABC}$ .  $\bar{\mathbf{P}}$ ,  $\bar{\mathbf{M}}$ , and  $\bar{\mathbf{n}}$  all have different directions when the deformation is finite.

The edges of the deformed tetrahedron are covariant base vectors  $\tilde{\mathbf{g}}_i$  that are tangent to deformed curvilinear material lines.

$$\tilde{\mathbf{g}}_i = \mathbf{e}_k \frac{\partial \bar{x}_k}{\partial x_i} \quad (39)$$

and

$$J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j} \quad (40)$$

The columns of  $\mathbf{J}$  are covariant base vectors  $\tilde{\mathbf{g}}_i$  that form non-orthogonal covariant basis. Contravariant base vectors  $\tilde{\mathbf{g}}^i$  are normal to the faces of the deformed tetrahedron formed by the covariant base vectors.

$$\tilde{\mathbf{g}}^j = \mathbf{e}_i \frac{\partial x_j}{\partial \bar{x}_i} \quad (41)$$

and

$$\bar{J}_{ij} = \frac{\partial x_i}{\partial \bar{x}_j} \quad (42)$$

The rows of  $\bar{\mathbf{J}}$  are contravariant base vectors  $\tilde{\mathbf{g}}^i$ . These form a non-orthogonal contravariant basis. Covariant and contravariant bases are reciprocal to each other [67].

#### 3.2.1. Contravariant and covariant Cauchy stress tensors

The definition of the stresses on the non-oblique faces of the deformed tetrahedron formed by the covariant base vectors  $\tilde{\mathbf{g}}_i$  in the contravariant directions orthogonal to the faces of the deformed tetrahedron is the most natural way. Let  $\tilde{\boldsymbol{\sigma}}^{(0)}$  or  $\tilde{\boldsymbol{\sigma}}^{(0)}$  be the contravariant stress tensor with components  $\tilde{\sigma}_{ij}^{(0)}$  or  $\tilde{\sigma}_{ij}^{(0)}$  with dyads  $\tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}_j$ . Component  $\tilde{\sigma}_{11}^{(0)}$  or  $\tilde{\sigma}_{11}^{(0)}$  is in the  $\tilde{\mathbf{g}}^1$  direction on a face of the tetrahedron with unit exterior normal  $\tilde{\mathbf{g}}^1$ , i.e. on the  $\tilde{\mathbf{g}}^1$  face. Likewise  $\tilde{\sigma}_{12}^{(0)}$  or  $\tilde{\sigma}_{12}^{(0)}$  and  $\tilde{\sigma}_{31}^{(0)}$  or  $\tilde{\sigma}_{31}^{(0)}$  act on  $\tilde{\mathbf{g}}^1$



and  $\tilde{\mathbf{g}}^3$  faces in the  $\tilde{\mathbf{g}}^2$  and  $\tilde{\mathbf{g}}^1$  directions. Using dyads  $\tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}_j$  or contravariant law of transformation, we can write

$$\boldsymbol{\sigma}^{(0)} = \tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}_j \sigma_{ij}^{(0)} \quad (43)$$

Using (39) in (43), we can write

$$\begin{aligned} \boldsymbol{\sigma}^{(0)} &= \mathbf{e}_i \otimes \mathbf{e}_j \sigma_{ij}^{(0)} \\ \sigma_{ij}^{(0)} &= J_{ik} \sigma_{kl}^{(0)} J_{jl} \\ [\boldsymbol{\sigma}^{(0)}] &= [J][\sigma^{(0)}][J]^T \end{aligned} \quad (44)$$

$\boldsymbol{\sigma}^{(0)}$  is a contravariant Cauchy stress tensor (Lagrangian description) from which  $\bar{\boldsymbol{\sigma}}^{(0)}$  can be easily obtained by replacing  $[J]$  by  $[\bar{J}]^{-1}$  and  $\boldsymbol{\sigma}^{(0)}$  by  $\bar{\boldsymbol{\sigma}}^{(0)}$  in (44). Since the dyads of  $\boldsymbol{\sigma}^{(0)}$  or  $\bar{\boldsymbol{\sigma}}^{(0)}$  are  $\mathbf{e}_i \otimes \mathbf{e}_j$ , the Cauchy principle holds between  $\bar{\mathbf{P}}$  and  $\bar{\boldsymbol{\sigma}}^{(0)}$ .

$$\bar{\mathbf{P}} = (\bar{\boldsymbol{\sigma}}^{(0)})^T \cdot \bar{\mathbf{n}} \quad (45)$$

Instead of using contravariant directions and stress components  $\boldsymbol{\sigma}^{(0)}$  and covariant basis  $\tilde{\mathbf{g}}_i$ , we could use covariant stress components  $(\sigma_{(0)})_{ij}$  or  $(\bar{\sigma}_{(0)})_{ij}$  and contravariant basis  $\tilde{\mathbf{g}}^i$ . Consideration of  $(\sigma_{(0)})_{ij}$  of course will require a different deformed tetrahedron such that covariant base vectors  $\tilde{\mathbf{g}}_i$  are normal to its oblique faces. The adverse consequences of choosing this measure of stress for finite deformation are discussed in reference [67]. Here we proceed with this measure as an alternative to the contravariant stress measure. Using dyads  $\tilde{\mathbf{g}}^i \otimes \tilde{\mathbf{g}}^j$  and components  $(\sigma_{(0)})_{ij}$ , we can write

$$\bar{\boldsymbol{\sigma}}_{(0)} = \tilde{\mathbf{g}}^i \otimes \tilde{\mathbf{g}}^j (\sigma_{(0)})_{ij} \quad (46)$$

And using (41)

$$\begin{aligned} \bar{\boldsymbol{\sigma}}_{(0)} &= \mathbf{e}_i \otimes \mathbf{e}_j (\bar{\sigma}_{(0)})_{ij} \\ (\bar{\sigma}_{(0)})_{ij} &= \bar{J}_{ki} (\sigma_{(0)})_{kl} \bar{J}_{lj} \\ [\bar{\sigma}_{(0)}] &= [\bar{J}]^T [\sigma_{(0)}] [\bar{J}] \end{aligned} \quad (47)$$

$\bar{\boldsymbol{\sigma}}_{(0)}$  is the covariant Cauchy stress tensor (Eulerian description) from which  $\boldsymbol{\sigma}_{(0)}$  can be obtained by replacing  $[\bar{J}]$  with  $[J]^{-1}$  and  $\bar{\boldsymbol{\sigma}}_{(0)}$  with  $\boldsymbol{\sigma}_{(0)}$  in (47). Since the dyads of  $\bar{\boldsymbol{\sigma}}_{(0)}$  are  $\mathbf{e}_i \otimes \mathbf{e}_j$ , the Cauchy principle holds between  $\bar{\mathbf{P}}$  and  $\bar{\boldsymbol{\sigma}}_{(0)}$ .

$$\bar{\mathbf{P}} = (\bar{\boldsymbol{\sigma}}_{(0)})^T \cdot \bar{\mathbf{n}} \quad (48)$$

### Remarks

The Cauchy stress tensors  $\boldsymbol{\sigma}^{(0)}$  or  $\bar{\boldsymbol{\sigma}}^{(0)}$  and  $\boldsymbol{\sigma}_{(0)}$  or  $\bar{\boldsymbol{\sigma}}_{(0)}$  are non-symmetric at this stage and so are the stress tensors  $\tilde{\boldsymbol{\sigma}}^{(0)}$  and  $\tilde{\boldsymbol{\sigma}}_{(0)}$ . Following the details in reference [67] we can also define Jaumann stress tensor  ${}^{(0)}\boldsymbol{\sigma}^J$  using  $\bar{\boldsymbol{\sigma}}^{(0)}$  and  $\bar{\boldsymbol{\sigma}}_{(0)}$  stress measures.

### 3.2.2. Contravariant and covariant Cauchy moment tensors

When the deformed tetrahedron with moment  $\bar{\mathbf{M}}$  on its oblique face  $\bar{A}\bar{B}\bar{C}$  is isolated from the volume  $\bar{V}$ , its faces will have existence of moments (per unit area) on them. As in case of stress measure, contravariant basis is the most natural way to define these. Following the notations parallel to those used in case of Cauchy stress tensors, we can write the following using contravariant measures of moment tensor.

$$\mathbf{m}^{(0)} = \tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}_j m_{ij}^{(0)} \quad (49)$$

Using (39) in (49) we obtain

$$\begin{aligned} \mathbf{m}^{(0)} &= \mathbf{e}_i \otimes \mathbf{e}_j m_{ij}^{(0)} \\ m_{ij}^{(0)} &= J_{ik} \tilde{m}_{kl}^{(0)} J_{jl} \\ [\mathbf{m}^{(0)}] &= [\mathbf{J}] [\tilde{\mathbf{m}}^{(0)}] [\mathbf{J}]^T \\ [\tilde{\mathbf{m}}^{(0)}] &= [\mathbf{J}]^{-1} [\mathbf{m}^{(0)}] [[\mathbf{J}]^{-1}]^T \end{aligned} \quad (50)$$

and the Cauchy principle

$$\bar{\mathbf{M}} = (\bar{\mathbf{m}}^{(0)})^T \cdot \bar{\mathbf{n}} \quad (51)$$

Likewise when using covariant measure of moment tensor we have

$$\bar{\mathbf{m}}_{(0)} = \tilde{\mathbf{g}}^i \otimes \tilde{\mathbf{g}}^j (\mathbf{m}_{(0)})_{ij} \quad (52)$$

And using (41) in (52) we obtain

$$\begin{aligned} \bar{\mathbf{m}}_{(0)} &= \mathbf{e}_i \otimes \mathbf{e}_j (\bar{\mathbf{n}}_{(0)})_{ij} \\ (\bar{\mathbf{m}}_{(0)})_{ij} &= \bar{J}_{kl} (\mathbf{m}_{(0)})_{kl} \bar{J}_{ij} \\ [\bar{\mathbf{m}}_{(0)}] &= [\bar{\mathbf{J}}]^T [\mathbf{m}_{(0)}] [\bar{\mathbf{J}}] \\ [\mathbf{m}_{(0)}] &= [[\bar{\mathbf{J}}]^{-1}]^T [\bar{\mathbf{m}}_{(0)}] [\bar{\mathbf{J}}]^{-1} \end{aligned} \quad (53)$$

and the Cauchy principle

$$\bar{\mathbf{M}} = (\bar{\mathbf{m}}_{(0)})^T \cdot \bar{\mathbf{n}} \quad (54)$$

As in case of stress tensors  $\bar{\boldsymbol{\sigma}}^{(0)}$  and  $\bar{\boldsymbol{\sigma}}_{(0)}$ , the moment tensors  $\bar{\mathbf{m}}^{(0)}$  and  $\bar{\mathbf{m}}_{(0)}$  are also non-symmetric at this stage.

## 4. Conservation and balance laws and the constitutive theories

We consider conservation and balance laws for non-classical solid continua as well as fluent continua based on the internal rotations and the Cosserat rotations as well as internal rotation rates and the Cosserat rotation rates respectively. The conservation and the balance laws as well as constitutive theories for non-classical continuum theories for solid and fluent continua based on internal rotations and the internal rotation rates can be easily extracted from the continuum theories derived using both rotations and rotation rates.

#### 4.1 Conservation and balance laws for solid continua incorporating internal and cosserat rotations

In the following we present conservation and balance laws for non-classical continuum theory based on internal and Cosserat rotations [68].

##### 4.1.1 Conservation of mass

The continuity equation resulting from the principle of conservation of mass remains the same for the non-classical continuum theory considered here as in case of classical continuum theory as long as the matter is treated homogeneous and isotropic. In Lagrangian description, continuity Equations [67, 69, 70] can be written as

$$\rho_0(\mathbf{x}) = |J|\rho(\mathbf{x}, t) \quad (55)$$

For infinitesimal deformation  $|J| \simeq 1$  hence  $\rho_0(\mathbf{x}) \simeq \rho(\mathbf{x}, t)$ , where  $\rho_0(\mathbf{x})$  is the density of the material point at  $\mathbf{x}$  in the reference configuration and  $\rho(\mathbf{x}, t)$  is the Lagrangian description of the density of a material point at  $\bar{\mathbf{x}}$  in the current configuration.

##### 4.1.2 Balance of linear momenta

For a deforming volume of matter the rate of change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation is same as that for classical continuum theory. Thus, we can write (for small deformation) the following [67,69,70]:

$$\begin{aligned} \rho_0 \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b - \nabla \cdot \boldsymbol{\sigma} &= 0 \\ \text{or} \quad \rho_0 \frac{D\{\mathbf{v}\}}{Dt} - \rho_0 \{\mathbf{F}^b\} - [\boldsymbol{\sigma}]^T \{\nabla\} &= 0 \end{aligned} \quad (56)$$

In Lagrangian description  $\frac{D}{Dt} = \frac{\partial}{\partial t}$  and  $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$  are velocities,  $\mathbf{F}^b$  are body forces per unit mass, and  $\boldsymbol{\sigma}$  is the Cauchy stress tensor. Equation (56) are momentum equations in  $x_1$ -,  $x_2$ -, and  $x_3$ -directions. The Cauchy stress tensor is nonsymmetric at this stage as its symmetry has not been established.

##### 4.1.3 Balance of angular momenta

The principle of balance of angular momenta for a non-classical continuum can be stated as follows: *The time rate of change of total moment of momenta for a non-classical continuum is equal to the vector sum of the moments of external forces and the moments.* Thus, due to the surface stress  $\bar{\mathbf{P}}$ , total surface moment  $\bar{\mathbf{M}}$  (per unit area) created when the internal and Cosserat rotations are resisted by the deforming continuum, body force  $\bar{\mathbf{F}}^b$  (per unit mass), and the momentum  $\bar{\rho}\bar{\mathbf{v}}d\bar{V}$  for an elemental mass  $\bar{\rho}d\bar{V}$  in the current configuration (using the Eulerian description) we can write the following:

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\rho}\bar{\mathbf{v}}d\bar{V} = \int_{\partial\bar{V}(t)} (\bar{\mathbf{x}} \times \bar{\mathbf{P}} - \bar{\mathbf{M}}) d\bar{A} + \int_{\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\rho}\bar{\mathbf{F}}^b d\bar{V} \quad (57)$$

Negative sign for  $\bar{\mathbf{M}}$  is due to clockwise rotations being positive. We consider each term in (57) individually. First consider

$$\begin{aligned}
\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{v}} d\bar{V} &= \frac{D}{Dt} \int_{V(t)} \epsilon_{ijk} \bar{x}_i \bar{v}_j \mathbf{e}_k \bar{\rho} d\bar{V} = \frac{D}{Dt} \int_V \epsilon_{ijk} \mathbf{e}_k x_i v_j \rho_0 dV \\
&= \int_V \rho_0 \epsilon_{ijk} \mathbf{e}_k \frac{D}{Dt} (x_i v_j) dV = \int_V \rho_0 \epsilon_{ijk} \mathbf{e}_k \left( v_i v_j + x_i \frac{Dv_j}{Dt} \right) dV \\
&= \int_V \rho_0 \epsilon_{ijk} \mathbf{e}_k x_i \frac{Dv_j}{Dt} dV
\end{aligned} \quad (58)$$

Consider the first term on the right hand side of (57):

$$\begin{aligned}
\int_{\partial \bar{V}(t)} (\bar{\mathbf{x}} \times \bar{\mathbf{P}} - \bar{\mathbf{M}}) d\bar{\mathbf{A}} &= \int_{\partial V(t)} [\bar{\mathbf{x}} \times (\bar{\boldsymbol{\sigma}})^T \cdot \bar{\mathbf{n}} - (\bar{\mathbf{m}})^T \cdot \bar{\mathbf{n}}] d\bar{\mathbf{A}} \\
&= \int_{\partial V} [\mathbf{x} \times (\boldsymbol{\sigma})^T \cdot \mathbf{n} - (\mathbf{m})^T \cdot \mathbf{n}] dA \\
&= \int_{\partial V} \mathbf{e}_k (\epsilon_{ijk} x_i \sigma_{mj} n_m - m_{mk} n_m) dA \\
&= \int_V \mathbf{e}_k [\epsilon_{ijk} (x_i \sigma_{mj})_{,m} - m_{mk,m}] dV \\
&= \int_V \mathbf{e}_k [\epsilon_{ijk} (\sigma_{ij} + x_i (\sigma_{mj})_{,m}) - m_{mk,m}] dV
\end{aligned} \quad (59)$$

in which  $\bar{\boldsymbol{\sigma}}$  is the *Cauchy stress tensor* and  $\bar{\mathbf{m}}$  is the *Cauchy moment tensor*. Next, consider the second term on the right hand side (57):

$$\int_{\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{F}}^b d\bar{V} = \int_{V(t)} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \bar{F}_j^b \bar{\rho} d\bar{V} = \int_V \mathbf{e}_k \epsilon_{ijk} x_i F_j^b \rho_0 dV \quad (60)$$

Substituting from (58), (59), and (60) into (57), we obtain

$$\int_V \mathbf{e}_k \epsilon_{ijk} \left[ x_i \left( \rho_0 \frac{Dv_j}{Dt} - \rho_0 F_j^b - \sigma_{mj,m} \right) \right] dV + \int_V \mathbf{e}_k (m_{mk,m} - \epsilon_{ijk} \sigma_{ij}) dV = 0 \quad (61)$$

Using balance of linear momenta (56) in (61), we arrive at

$$\int_V \mathbf{e}_k (m_{mk,m} - \epsilon_{ijk} \sigma_{ij}) dV = 0 \quad (62)$$

and, since the volume  $V$  is arbitrary, we have

$$m_{mk,m} - \epsilon_{ijk} \sigma_{ij} = 0 \quad (63)$$

$$\text{or} \quad \nabla \cdot \mathbf{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \quad (64)$$

Equation (63) represents balance of angular momenta. *The Cauchy stress tensor  $\boldsymbol{\sigma}$  is non-symmetric.* From (63) one notes that *antisymmetric components of the Cauchy stress tensor  $\boldsymbol{\sigma}$  are balanced by the gradients of the Cauchy moment tensor.*

## Remarks

- (a) In the balance of angular momenta, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces. Thus, this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components do not contribute to this. Hence, the moments contained in this balance law due to stresses are only caused by the shear stresses contained in the skew-symmetric part of the Cauchy stress tensor.
- (b) In the case of classical continuum theory, the balance of angular momenta is a statement of self equilibrating moments due to the symmetry of shear stresses

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} = 0 \quad (65)$$

An important point to note is that (65) is a result of stress couples due to shear stresses.

- (c) In the case of non-classical continua, the existence of moments  $\mathbf{m}$  due to the material constitution resisting the rotations, both internal and Cosserat rotations, results in the shear stress couples from the antisymmetric part of the Cauchy stress tensor being balanced by the internal moments. Thus, for non-classical continua, the balance of angular momenta yields (64) instead of (65).
- (d) Both the non-classical and classical continuum theories use stress couples in the balance of angular momenta.
- (e) From (63) it is clear that gradients of  $\mathbf{m}$  equilibrate only with the antisymmetric (shear) components of the stress tensor  $\boldsymbol{\sigma}$ .
- (f) Lastly, the present study demonstrates that the varying rotations, both internal and Cosserat, at the neighboring material points when resisted by the deforming matter require existence of internal moment tensor  $\mathbf{m}$ . The balance of angular momenta establishes a relationship between  $\mathbf{m}$  and  $\boldsymbol{\sigma}$ .

### 4.1.4 Balance of moment of moments: new balance law

Need for this balance law for non-classical solid and fluent continua and a complete derivation based on rate consideration can be found in references [71,72] for solid and fluent continua. In this section we consider this balance law for solid continua [71].

Yang, et al. [66] presented the following reasoning for consideration of additional requirements advocated to be necessary for equilibrium of deforming solid matter in couple stress non-classical continuum theories.

*When a system of forces is applied to a system of multiple particles, the equilibrium relations are derived from a resultant force and a resultant couple of forces applied to an arbitrary point. The moment of a couple of forces is a free vector in classical mechanics, which means that the effect of the couple applied at an arbitrary point in the space of the system of materials particles is independent of the position of the point. In other words, the couple can translate to any point in the space freely and the resulting effects are unchanged. As a result, only the conventional force equilibrium and the moment equilibrium (balance of linear and angular momenta) are involved in the equilibrium relations. Equivalence of a couple resulting from the rotations  $\boldsymbol{\theta}$  that is not a free vector but a driving force that rotates the material particles requires considerations (see [66] for details) that eventually results into balance of moment of moments or couples for static equilibrium.*

In this reasoning at the onset of the derivation, the moment of the moments due to the antisymmetric part of the Cauchy stress tensor and the moment of the moment  $\bar{\mathbf{M}}$  acting on the oblique plane of the deformed tetrahedron are assumed to equilibrate. This is termed by the authors as *balance of moment of moments balance law*. In this approach there are two major points to be clarified: (i) Is this a balance law, and (ii) if what Yang, et al. [66] presented is based on static considerations, then does it ensure dynamic equilibrium of the deforming non-classical solid continua during evolution? Based on the definition of balance laws in continuum mechanics (balance of linear and angular momenta, for example), a balance law must be based on rate considerations. The work of Yang, et al. [66] as presented by them is static equilibrium, hence it is perhaps more appropriate to label this as an equilibrium consideration at this stage rather than a balance law.

First, we consider inductive reasoning to demonstrate and establish why there is a need for an additional law in non-classical continuum theories to ensure dynamic equilibrium of the deforming solid continua in the presence of internal rotations and conjugate moments. In classical continuum theories for solid continua that consider displacements as the only observable quantities at the material points and their conjugate forces (or stresses), it is well known that the balance of linear momenta and the balance of angular momenta must hold for the dynamic equilibrium of the deforming solid continua. That is, the rate of change of linear momenta must be balanced by body forces and the average stress  $\bar{\mathbf{P}}$  on the oblique plane of the tetrahedron for any arbitrary volume of matter (balance of linear momenta). The rate of change of the moment of linear momenta must be balanced by the moment of the body forces and the moment of average stress  $\bar{\mathbf{P}}$  on the oblique plane of the tetrahedron (balance of angular momenta balance law). These two balance laws ensure stable dynamic equilibrium of the deforming volume of solid continua in classical continuum mechanics at any instant of time. Thus, we note that when the displacements are the only kinematic variables, two balance laws are required. The first balance law is the dynamic balance of the quantities conjugate to displacements that are forces, the balance of linear momenta, and the second one is the dynamic balance of the moments of the quantities conjugate to the displacements, i.e., moments of forces, the balance of angular momenta.

### Remarks

- (1) We note that the balances of linear and angular momenta contain physics purely related to the forces and the moments due to the forces.
- (2) When rotations and their conjugate moments are introduced, the balance of linear momenta (purely related to the forces) remains unchanged as the nonsymmetry of the Cauchy stress tensor is also present in classical theories until the balance of angular momenta establishes it to be symmetric.
- (3) Additional moments introduced by the consideration of internal rotations must now be considered to modify the dynamic balance of moments, i.e., the balance of angular momenta used in classical theories. The end result is the relationship between additional conjugate quantities introduced due to non-classical theories, i.e., the antisymmetric components of the Cauchy stress tensor and the Cauchy moment tensor. We note that neither of these exist in the classical continuum theory. Thus, due to internal

rotations and the conjugate moments, the first balance law needed is the balance of angular momenta. This already exists due to the classical theory, hence it is modified due to the presence of additional moments conjugate to the internal rotations and also due to the antisymmetric components of the Cauchy stress tensor.

- (4) A new balance law, *the law of balance of moment of moments* (parallel to the balance of angular momenta in classical theories), is required for dynamic equilibrium in the presence of internal rotations, their rates, and conjugate moments. This balance law must be a rate law just like all other balance laws, and must only contain the physics related to the non-classical behavior, i.e., possibly rotations and their rates, the conjugate Cauchy moment tensor, and the antisymmetric part of the Cauchy stress tensor. Thus, in the derivation of this balance law we must consider the rate of the moment of angular momenta only due to internal rotation rates to balance with: (i) the moment of moments of those components associated with  $\bar{\mathbf{P}}$  that are only related to non-classical physics, i.e., the antisymmetric components of the Cauchy stress tensor, and (ii) the moment of  $\bar{\mathbf{M}}$  which is only due to non-classical physics.
- (5) We remark that consideration of the following as a balance law in the non-classical theory considered here is invalid.

$$\frac{D}{Dt} \int_{V(t)} \bar{\mathbf{x}} \times (\bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{v}}) d\bar{V} = \int_{\partial V(t)} \bar{\mathbf{x}} \times (\bar{\mathbf{x}} \times \bar{\mathbf{P}} - \bar{\mathbf{M}}) d\bar{\mathbf{A}} + \int_{V(t)} \bar{\mathbf{x}} \times (\bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{F}}^b) d\bar{V} \quad (66)$$

- (a) The left-hand side is purely due to the classical continuum physics, hence cannot be part of this balance law. (b)  $\int_{\partial V(t)} \bar{\mathbf{x}} \times (\bar{\mathbf{x}} \times \bar{\mathbf{P}}) d\bar{\mathbf{A}}$  is invalid as it contains the symmetric

part of the Cauchy stress tensor (after application of the Cauchy principle) which is also part of the classical continuum theory. In this expression only antisymmetric components of the Cauchy stress tensor should be considered as these are the only components related to non-classical behavior.

(c)  $\int_{V(t)} \bar{\mathbf{x}} \times (\bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{F}}^b) d\bar{V}$  is also purely due to

classical continuum physics, hence cannot be considered in this balance law. Presence

of  $\int_{\partial V(t)} (\bar{\mathbf{x}} \times \bar{\mathbf{M}}) d\bar{\mathbf{A}}$  is valid as  $\bar{\mathbf{M}}$  is purely due to internal rotation physics.

- (6) The derivation using (66) leads to erroneous results, as expected due to the fact that (66) mostly contains physics that is purely related to the classical continuum theory (except  $\bar{\mathbf{M}}$ ) that should be eliminated from this balance law as this balance law is only necessitated due to new physics related to internal rotations.
- (7) We note that a kinematic variable requires two balance laws: (i) the first is related to the dynamic balance of the quantity conjugate to the kinematic variable and (ii) the second one is related to the dynamic balance of the moment of the quantities conjugate to the kinematic variable. Displacements as kinematic variables need the balance of linear and angular momenta, which are dynamic balances of forces and their moments. Introduction of internal rotations and their rates require dynamic balance of moments (which already exists as the balance of angular momenta) and dynamic balance of moment of moments, a new balance law. Thus, for each new

kinematic variable we need to consider (i) the dynamic balance of its conjugate quantity that already exists from the previous kinematic variable, hence can be modified to accommodate the influence of new physics, and (ii) dynamic balance of the moment of its conjugate quantity related only to the physics associated with the new kinematic variable, which is a new balance law that needs to be derived using rate considerations. This inductive reasoning holds for the introduction of each new kinematic variable.

- (8) In the next section we present derivation of the *law of balance of moment of moments* based on rate considerations.
- (9) Since all derivations initiate in the deformed configuration, it is also true for the derivation presented here is also applicable to fluent continua.

In this balance law [71,72] we must consider the rate of moment of angular momenta due to rotation rates to balance with the moment of moments of the antisymmetric components of the Cauchy stress tensor and the moments of  $\bar{\mathbf{M}}$ , all of which are only related to the non-classical physics due to internal rotations and the associated conjugate moments. We can write

$$\left( \begin{array}{c} \text{rate of moment of the} \\ \text{angular moment a due to} \\ \text{internal rotation rates} \\ \text{over } \bar{V}(t) \end{array} \right) = \left( \begin{array}{c} \text{moment of moments due} \\ \text{to antisymmetric components} \\ \text{of the Cauchy stress} \\ \text{tensor over } \bar{V}(t) \end{array} \right) - \left( \begin{array}{c} \text{moment of } \bar{\mathbf{M}} \\ \text{over } \partial \bar{V}(t) \end{array} \right) \quad (67)$$

The negative sign is due to the assumption of clockwise internal rotations to be positive, hence the corresponding moment tensor must be positive in the same sense. We note that in continuum theories for continuous media we assume that the material points or particles have mass but no dimensions, thus the angular momenta associated with the material particles due to rotation rates are zero. Thus, (67) reduces to

$$\left( \begin{array}{c} \text{moment of moments due} \\ \text{to antisymmetric components} \\ \text{of the Cauchy stress} \\ \text{tensor over } \bar{V}(t) \end{array} \right) - \left( \begin{array}{c} \text{moment of } \bar{\mathbf{M}} \\ \text{over } \partial \bar{V}(t) \end{array} \right) = 0 \quad (68)$$

If we consider the current configuration at time  $t$ , then in the Eulerian description we can write (68) as

$$\int_{\bar{V}(t)} \bar{\mathbf{x}} \times (\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}) d\bar{V} - \int_{\partial \bar{V}(t)} \bar{\mathbf{x}} \times \bar{\mathbf{M}} d\bar{A} = 0 \quad (69)$$

### Remarks

- (1) In the absence of (67), a rate statement, if we consider (68) and (69) directly, then we could mistakenly view (68) and (69) as equilibrium of moment of moments, a static consideration. This is obviously incorrect. As stated by Yang, et al. [66], (69) indeed is a balance law, even though its derivation based on a statement like (67) is not reported in Reference [66]. Due to the fact that the left-hand side of (67) is zero,



the balance law (67) results in (69), which unfortunately has the appearance of an equilibrium statement.

- (2) Henceforth, in this paper we refer to (69) as *the law of balance of moment of moments*. For the non-classical physics considered in this paper, the balance law (67) reduces to (69), which can also be labeled as the equilibrium of moment of moments due to the absence of rate terms (as they are zero). Thus, we can also say *the law of balance/equilibrium of moment of moments*.

We expand the second term in (69) and then convert the integral over  $\partial\bar{V}$  to the integral over  $\bar{V}$  using the divergence theorem:

$$\begin{aligned}
 \int_{\partial\bar{V}} \bar{\mathbf{x}} \times \bar{\mathbf{M}} \, d\bar{A} &= \int_{\partial\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \bar{M}_j \, d\bar{A} \\
 &= \int_{\partial\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \bar{m}_{mj} \bar{n}_m \, d\bar{A} \\
 &= \int_{\bar{V}} \mathbf{e}_k (\epsilon_{ijk} \bar{x}_i \bar{m}_{mj})_{,m} \, d\bar{V} \\
 &= \int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} (\bar{m}_{ij} + \bar{x}_i \bar{m}_{mj,m}) \, d\bar{V} \\
 &= \int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} + \int_{\bar{V}} \bar{\mathbf{x}} \times (\bar{\nabla} \cdot \bar{\mathbf{m}}) \, d\bar{V}
 \end{aligned} \tag{70}$$

Using Equation (70) in (69) and collecting terms:

$$\int_{\bar{V}} \bar{\mathbf{x}} \times (-\bar{\nabla} \cdot \bar{\mathbf{m}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}) \, d\bar{V} - \int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0 \tag{71}$$

The first term in (71) vanishes due to the balance of angular momenta, giving the condition

$$\int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0 \tag{72}$$

which, because  $\bar{V}$  is arbitrary, yields

$$\epsilon_{ijk} \bar{m}_{ij} = 0 \quad \text{and} \quad \epsilon_{ijk} m_{ij} = 0 \tag{73}$$

Equation (73) implies that Cauchy moment tensor  $\mathbf{m}$  is symmetric in non-classical continuum theories when law of balance of moment of moments is used as an additional balance law.

### Remarks

- (1) We note that  $\mathbf{m}$  in balance of angular momenta (64) was nonsymmetric (section 4.1.3), but now it is symmetric due to consideration of law of balance of moment of moments.
- (2) In the energy equation as well as entropy inequality  $\mathbf{m}$  will be symmetric as well.

#### 4.1.5 First law of thermodynamics

The sum of work and heat added to a deforming volume of matter must result in increase of the energy of the system. This is expressed as a rate equation in the Eulerian description as

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \quad (74)$$

where  $\bar{E}_t$ ,  $\bar{Q}$  and  $\bar{W}$  are total energy, heat added, and work done. We have

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt} \int_{V(t)} \bar{\rho} \left( \bar{e} + \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} - \bar{\mathbf{F}}^b \cdot \bar{\mathbf{u}} \right) d\bar{V} \quad (75)$$

$$\frac{D\bar{Q}}{Dt} = - \int_{\partial V(t)} \bar{\mathbf{q}} \cdot \bar{\mathbf{n}} d\bar{A} \quad (76)$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial V(t)} (\bar{\mathbf{P}} \cdot \bar{\mathbf{v}} + \bar{\mathbf{M}} \cdot {}_t\dot{\bar{\boldsymbol{\Theta}}}) d\bar{A} \quad (77)$$

Here  $\bar{e}$  is specific internal energy,  $\bar{\mathbf{F}}^b$  is body force vector per unit mass,  $\bar{\mathbf{q}}$  is rate of heat.

Note that the additional term  $\bar{\mathbf{M}} \cdot {}_t\dot{\bar{\boldsymbol{\Theta}}}$  in  $\frac{D\bar{W}}{Dt}$  contributes additional rate of work due to rates of total rotations. Expanding integrals and following reference [67], one can show that

$$\frac{D}{Dt} \int_{V(t)} \bar{\rho} \left( \bar{e} + \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} - \bar{\mathbf{F}}^b \cdot \bar{\mathbf{u}} \right) d\bar{V} = \int_V \left( \rho_0 \frac{De}{Dt} + \rho_0 \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b \cdot \mathbf{v} \right) dV \quad (78)$$

Using

$$\begin{aligned} \bar{\mathbf{q}} \cdot \bar{\mathbf{n}} d\bar{A} &= \mathbf{q} \cdot \mathbf{n} dA \\ \bar{\rho} d\bar{V} &= \rho_0 dV \\ d\bar{V} &= |J| dV \end{aligned} \quad (79)$$

and applying the divergence theorem, we obtain

$$- \int_{\partial V(t)} \bar{\mathbf{q}} \cdot \bar{\mathbf{n}} d\bar{A} = - \int_{\partial V} \mathbf{q} \cdot \mathbf{n} dA = - \int_V \nabla \cdot \mathbf{q} dV \quad (80)$$

Using stress tensor  $\boldsymbol{\sigma}$  and moment tensor  $\mathbf{m}$ , and following reference [67], one can show that

$$\bar{\mathbf{P}} \cdot \bar{\mathbf{v}} d\bar{A} = \mathbf{v} \cdot (\boldsymbol{\sigma})^T \cdot \mathbf{n} dA = \left( \mathbf{v} \cdot (\boldsymbol{\sigma})^T \right) \cdot d\mathbf{A} \quad (81)$$

$$\bar{\mathbf{M}} \cdot {}_t\dot{\bar{\boldsymbol{\Theta}}} d\bar{A} = \left( {}_t\dot{\boldsymbol{\Theta}} \cdot (\mathbf{m})^T \right) \cdot \mathbf{n} dA = \left( {}_t\dot{\boldsymbol{\Theta}} \cdot (\mathbf{m})^T \right) \cdot d\mathbf{A} \quad (82)$$

Thus, one can write the following for (74):

$$\begin{aligned}
& \int_V \left( \rho_0 \frac{De}{Dt} + \rho_0 \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b \cdot \mathbf{v} \right) dV \\
&= - \int_V \nabla \cdot \mathbf{q} dV + \int_{\partial V} \left( \mathbf{v} \cdot (\boldsymbol{\sigma})^T \right) \cdot d\mathbf{A} + \int_{\partial V} \left( {}_t\dot{\boldsymbol{\theta}} \cdot (\mathbf{m})^T \right) \cdot d\mathbf{A} \\
&= - \int_V \nabla \cdot \mathbf{q} dV + \int_V \nabla \cdot \left( \mathbf{v} \cdot (\boldsymbol{\sigma})^T \right) dV + \int_V \nabla \cdot \left( {}_t\dot{\boldsymbol{\theta}} \cdot (\mathbf{m})^T \right) dV
\end{aligned} \quad (83)$$

Following reference [67] one can also show that

$$\nabla \cdot \left( \mathbf{v} \cdot (\boldsymbol{\sigma})^T \right) = \mathbf{v} \cdot (\nabla \cdot \boldsymbol{\sigma}) + \sigma_{ji} \frac{\partial v_i}{\partial x_j} \quad (84)$$

$$\nabla \cdot \left( {}_t\dot{\boldsymbol{\theta}} \cdot (\mathbf{m})^T \right) = {}_t\dot{\boldsymbol{\theta}} \cdot (\nabla \cdot \mathbf{m}) + m_{ji} \frac{\partial \dot{\theta}_i}{\partial x_j} \quad (85)$$

and substituting from (84) and (85) into (83)

$$\begin{aligned}
& \int_V \left( \rho_0 \frac{De}{Dt} + \rho_0 \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b \cdot \mathbf{v} \right) dV \\
&= - \int_V \nabla \cdot \mathbf{q} dV + \int_V \left( \mathbf{v} \cdot (\nabla \cdot \boldsymbol{\sigma}) + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + {}_t\dot{\boldsymbol{\theta}} \cdot (\nabla \cdot \mathbf{m}) + m_{ji} \frac{\partial \dot{\theta}_i}{\partial x_j} \right) dV
\end{aligned} \quad (86)$$

or

$$\begin{aligned}
& \int_V \mathbf{v} \cdot \left( \rho_0 \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b - \nabla \cdot \boldsymbol{\sigma} \right) dV \\
&+ \int_V \left( \rho_0 \frac{De}{Dt} + \nabla \cdot \mathbf{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\theta}_i}{\partial x_j} - {}_t\dot{\boldsymbol{\theta}} \cdot (\nabla \cdot \mathbf{m}) \right) dV = 0
\end{aligned} \quad (87)$$

Using (56) (balance of linear momenta), (87) reduces to

$$\int_V \left( \rho_0 \frac{De}{Dt} + \nabla \cdot \mathbf{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\theta}_i}{\partial x_j} - {}_t\dot{\boldsymbol{\theta}} \cdot (\nabla \cdot \mathbf{m}) \right) dV = 0 \quad (88)$$

Since volume  $V$  is arbitrary, the following holds:

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \mathbf{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - \left( m_{ji} \frac{\partial \dot{\theta}_i}{\partial x_j} + {}_t\dot{\boldsymbol{\theta}} \cdot (\nabla \cdot \mathbf{m}) \right) = 0 \quad (89)$$

We note that in  ${}_t\dot{\boldsymbol{\theta}} \cdot (\nabla \cdot \mathbf{m})$ , the term  $\nabla \cdot \mathbf{m}$  can be substituted from (64) thereby eliminating gradients of  $\mathbf{m}$  but introducing  $\boldsymbol{\sigma}$  in its place.

#### 4.1.6 Second law of thermodynamics

If  $\bar{\eta}$  is the entropy density in volume  $\bar{V}(t)$ ,  $\bar{h}$  is the entropy flux between  $\bar{V}(t)$  and the volume of matter surrounding it, and  $\bar{s}$  is the source of entropy in  $\bar{V}(t)$  due to non-

contacting bodies, then the rate of increase of entropy in volume  $\bar{V}(t)$  is at least equal to that supplied to  $\bar{V}(t)$  from all contacting and non-contacting sources [67]. Thus

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} d\bar{V} \geq \int_{\partial \bar{V}(t)} \bar{h} d\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} d\bar{V} \quad (90)$$

Using Cauchy's postulate for  $\bar{h}$

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\mathbf{n}} \quad (91)$$

Using (91) in (90)

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} d\bar{V} \geq - \int_{\partial \bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\mathbf{n}} d\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} d\bar{V} \quad (92)$$

Inequality (92) needs to be transformed to Lagrangian description. This can be done using

$$\begin{aligned} d\bar{V} &= |J| dV \\ \bar{\rho}_0 &= |J| \bar{\rho} \\ \bar{\boldsymbol{\psi}} \cdot \bar{\mathbf{n}} d\bar{A} &= \boldsymbol{\psi} \cdot \mathbf{n} dA \end{aligned} \quad (93)$$

Using (93) in (92)

$$\frac{D}{Dt} \int_V \eta \rho_0 dV \geq - \int_{\partial V} \boldsymbol{\psi} \cdot \mathbf{n} dA + \int_V s \rho_0 dV \quad (94)$$

Using Gauss's divergence theorem for the terms over  $\partial V$  gives (noting that  $\boldsymbol{\psi}$  is a tensor of rank one)

$$\frac{D}{Dt} \int_V \eta \rho_0 dV \geq - \int_V \nabla \cdot \boldsymbol{\psi} dV + \int_V s \rho_0 dV \quad (95)$$

or

$$\int_V \left( \rho_0 \frac{D\eta}{Dt} + \nabla \cdot \boldsymbol{\psi} - \rho_0 s \right) dV \geq 0 \quad (96)$$

and since volume  $V$  is arbitrary

$$\rho_0 \frac{D\eta}{Dt} + \nabla \cdot \boldsymbol{\psi} - \rho_0 s \geq 0 \quad (97)$$

Inequality (97) is the entropy inequality and is the most fundamental form resulting from the second law of thermodynamics. Inequality (97) is strictly a statement that contains entropy terms, hence contains no information regarding reversible deformation processes such as in case of elastic solids, thus provides no information or mechanisms regarding the derivations of the constitutive theories for such solids. Only when the mechanical rate of work results in rate of entropy production will inequality (97) have some information regarding the associated conjugate pairs that result in rate of entropy production. One can also note that (97) in its present form also does not provide any information regarding the constitutive theory for heat vector  $\mathbf{q}$ .

Another form of the entropy inequality is possible using relationship between  $\psi$  and  $\mathbf{q}$  and the energy equation. Since the energy equation has all possible mechanisms that result in energy storage and dissipation, this form of the entropy inequality derived using energy equation is expected to be helpful in the derivations of the constitutive theories. Using

$$\psi = \frac{\mathbf{q}}{\theta}, \quad s = \frac{r}{\theta} \quad (98)$$

where  $\theta$  is absolute temperature,  $\mathbf{q}$  is the heat vector and  $r$  is a suitable potential, then

$$\nabla \cdot \psi = \psi_{i,i} = \frac{q_{i,i}}{\theta} - \frac{q_i \theta_{,i}}{\theta^2} = \frac{q_{i,i}}{\theta} - \frac{q_i g_i}{\theta^2} = \frac{\nabla \cdot \mathbf{q}}{\theta} - \frac{\mathbf{q} \cdot \mathbf{g}}{\theta^2} \quad (99)$$

Substituting from (99) into (97) and multiplying throughout by  $\theta$  yields

$$\rho_0 \theta \frac{D\eta}{Dt} + \left( \nabla \cdot \mathbf{q} - \rho_0 r \right) - \frac{1}{\theta} \mathbf{q} \cdot \mathbf{g} \geq 0 \quad (100)$$

From energy Equation (89) (after inserting  $\rho_0 r$  term)

$$\nabla \cdot \mathbf{q} - \rho_0 r = -\rho_0 \frac{De}{Dt} + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + m_{ji} \frac{\partial_t \dot{\theta}_i}{\partial x_j} + {}_t \dot{\theta} \cdot (\nabla \cdot \mathbf{m}) \quad (101)$$

Substituting from (101) into (100)

$$\rho_0 \theta \frac{D\eta}{Dt} - \rho_0 \frac{De}{Dt} + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + m_{ji} \frac{\partial_t \dot{\theta}_i}{\partial x_j} + {}_t \dot{\theta} \cdot (\nabla \cdot \mathbf{m}) - \frac{1}{\theta} \mathbf{q} \cdot \mathbf{g} \geq 0 \quad (102)$$

or

$$\rho_0 \left( \frac{De}{Dt} - \theta \frac{D\eta}{Dt} \right) + \frac{1}{\theta} \mathbf{q} \cdot \mathbf{g} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial_t \dot{\theta}_i}{\partial x_j} - {}_t \dot{\theta} \cdot (\nabla \cdot \mathbf{m}) \leq 0 \quad (103)$$

Let  $\Phi$  be the Helmholtz free energy density defined by

$$\Phi = e - \eta \theta \quad (104)$$

Then we have

$$\frac{De}{Dt} - \theta \frac{D\eta}{Dt} = \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \quad (105)$$

Substituting from (105) into (103) gives

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial_t \dot{\theta}_i}{\partial x_j} - {}_t \dot{\theta} \cdot (\nabla \cdot \mathbf{m}) \leq 0 \quad (106)$$

or

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr} \left( [\sigma] \left[ \frac{\partial \{v\}}{\partial \{x\}} \right] \right) - \text{tr} \left( [m] \left[ \frac{\partial \{ {}_t \dot{\theta} \}}{\partial \{x\}} \right] \right) - {}_t \dot{\theta} \cdot (\nabla \cdot \mathbf{m}) \leq 0 \quad (107)$$

or

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr} \left( [\sigma] [{}^d\dot{J}] \right) - \text{tr} \left( [m] [{}^\circ\dot{J}^t] \right) - {}_t\dot{\Theta} \cdot (\nabla \cdot \mathbf{m}) \leq 0 \quad (108)$$

in which  $[{}^\circ\dot{J}^t]$  is the material derivative of  $[{}^\circ J^t]$  defined in (14) and  $[{}^d\dot{J}]$  is the material derivative of  $[{}^d J]$ , which is same as  $[\dot{J}]$  as  $[\dot{J}] = [{}^d\dot{J}]$ , however it is beneficial to work with  $[{}^d\dot{J}]$  and  $[{}^d J]$  for small deformation or small strain case.

Inequality (108) is the most fundamental form of the entropy inequality in Helmholtz free energy density  $\Phi$ . A slightly more expanded and more useful form of (108) in deriving constitutive theories can be obtained using (9).

$$\begin{aligned} [\mathbb{J}] &= [{}_s^d J] + [{}_a^d J] - [{}_a^e \dot{V}] = [{}^d J] - [{}_a^e \dot{V}] = [\dot{\epsilon}] + [{}_a^t \dot{r}] \\ [\dot{\mathbb{J}}] &= [{}_s^d \dot{J}] + [{}_a^d \dot{J}] - [{}_a^e \dot{\dot{V}}] = [{}^d \dot{J}] - [{}_a^e \dot{\dot{V}}] = [\dot{\epsilon}] + [{}_a^t \dot{r}] \end{aligned} \quad (109)$$

or

$$[{}^d J] = [\mathbb{J}] + [{}_a^e \dot{V}] \quad (110)$$

Taking material derivatives of both sides, we obtain

$$[{}^d \dot{J}] = [\dot{\mathbb{J}}] + [{}_a^e \dot{\dot{V}}] = [\dot{J}] \quad (111)$$

Also, using (14) and (12)

$$[{}^\circ J^t] = \left[ \frac{\partial \{ {}_t \Theta \}}{\partial \{ x \}} \right] = \left[ \frac{\partial \{ {}_t \Theta \}}{\partial \{ x \}} \right] - \left[ \frac{\partial \{ {}_e \Theta \}}{\partial \{ x \}} \right] = [{}^\circ J^t] - [{}^\circ J^e] \quad (112)$$

Hence

$$[{}^\circ \dot{J}^t] = [{}^\circ \dot{J}^t] - [{}^\circ \dot{J}^e] \quad (113)$$

and from balance of angular momenta (63) becomes

$$\nabla \cdot \mathbf{m} = \boldsymbol{\epsilon} : \boldsymbol{\sigma} \quad (114)$$

and

$${}_t\dot{\Theta} = {}_i\dot{\Theta} - {}_e\dot{\Theta} \quad (115)$$

Substituting from (111), (114), and (115) into (108), we arrive at

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr} \left( [\sigma] ([\dot{\mathbb{J}}] + [{}_a^e \dot{\dot{V}}]) \right) - \text{tr} \left( [m] [{}^\circ \dot{J}^t] \right) - ({}_i\dot{\Theta} - {}_e\dot{\Theta}) \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \quad (116)$$

or

$$\begin{aligned} \rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr} \left( [\sigma] [\dot{\mathbb{J}}] \right) - \text{tr} \left( [\sigma] [{}_a^e \dot{\dot{V}}] \right) - \text{tr} \left( [m] [{}^\circ \dot{J}^t] \right) - {}_i\dot{\Theta} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) + {}_e\dot{\Theta} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \\ \leq 0 \end{aligned} \quad (117)$$

A simple calculation shows that

$$\text{tr}([\sigma][{}^e_a\dot{\mathbb{Y}}]) = {}_e\dot{\Theta} \cdot (\epsilon : \sigma) \quad (118)$$

Using (118) in (117), (117) reduces to

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr}([\sigma][\dot{\mathbb{J}}]) - \text{tr}([m][{}^\theta J^t]) - {}_i\dot{\Theta} \cdot (\epsilon : \sigma) \leq 0 \quad (119)$$

The entropy inequality (119) is the desired form that is useful in deriving constitutive theories. By making similar substitutions and simplifications, the energy Equation (89) can be written as follows:

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \mathbf{q} - \text{tr}([\sigma][\dot{\mathbb{J}}]) - \text{tr}([m][{}^\theta J^t]) - {}_i\dot{\Theta} \cdot (\epsilon : \sigma) = 0 \quad (120)$$

#### 4.1.7 Rate of work conjugate pairs in the entropy inequality

Once the rate of work conjugate pairs are established from the energy equation or entropy inequality, the constitutive theories can also be derived using these in conjunction with the representation theorem or theory of generators and invariants. Details are considered in the following. We note that  $[\sigma]$ ,  $[\mathbb{J}]$  are both nonsymmetric tensors whereas  $[m]$  is a symmetric tensor (when balance of moment of moments is used as a balance law, as we have done in this paper) but  $[{}^\theta J^t]$  is a nonsymmetric tensor. Whether  $([\sigma], [\dot{\mathbb{J}}])$  and  $([m], [{}^\theta J^t])$  are rate of work conjugate pairs or not needs to be established.

Consider entropy inequality (119). Decompose  $\sigma$  into symmetric ( ${}_s\sigma$ ) and antisymmetric ( ${}_a\sigma$ ) tensors and use  $\dot{\mathbb{J}}$  from (109). Also decompose  ${}^\theta J^t$  into symmetric ( ${}_s{}^\theta J^t$ ) and antisymmetric ( ${}_a{}^\theta J^t$ ) tensors and substitute these into the entropy inequality (119).

$$\sigma = {}_s\sigma + {}_a\sigma \quad (121)$$

$${}^\theta J^t = {}_s{}^\theta J^t + {}_a{}^\theta J^t \quad (122)$$

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr}([{}_s\sigma][{}_s\dot{\mathbb{J}}] + [{}_a\sigma][{}_a\dot{\mathbb{J}}]) - \text{tr}([m]([{}_s{}^\theta J^t] + [{}_a{}^\theta J^t])) - {}_i\dot{\Theta} \cdot (\epsilon : \sigma) \leq 0 \quad (123)$$

Since

$$\text{tr}([{}_s\sigma][{}_a\dot{\mathbb{J}}]) = 0 \quad (124)$$

$$\text{tr}([{}_a\sigma][{}_s\dot{\mathbb{J}}]) = 0 \quad (125)$$

$$\text{tr}([m][{}_a{}^\theta J^t]) = 0 \quad (126)$$

entropy inequality (123) can be written as

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr}([{}_s\sigma][\dot{\epsilon}]) - \text{tr}([{}_a\sigma][{}_a^t\dot{r}]) - \text{tr}([m][{}_s^{\theta}\dot{J}^t]) - {}_i\dot{\Theta} \cdot (\epsilon : \sigma) \leq 0 \quad (127)$$

In (127)  $\mathbf{q}$  and  $\mathbf{g}$ ,  $[{}_s\sigma]$  and  $[\dot{\epsilon}]$ ,  $[{}_a\sigma]$  and  $[{}_a^t\dot{r}]$ ,  $[m]$  and  $[{}_s^{\theta}\dot{J}^t]$  are the rate of work conjugate pairs that are in conformity with the works of Spencer, Wang, Zheng, etc [73–89], i.e. a symmetric tensor is conjugate with a symmetric tensor and an antisymmetric tensor is conjugate with an antisymmetric tensor. Using (121), (122), and (124) – (126), the energy equation can also be written as

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \mathbf{q} - \text{tr}([{}_s\sigma][\dot{\epsilon}]) - \text{tr}([{}_a\sigma][{}_a^t\dot{r}]) - \text{tr}([m][{}_s^{\theta}\dot{J}^t]) - {}_i\dot{\Theta} \cdot (\epsilon : \sigma) = 0 \quad (128)$$

From the conjugate pairs it is straightforward to conclude that for the simplest possible case

$$\begin{aligned} {}_s\sigma &= {}_s\sigma(\epsilon, \theta) \\ {}_a\sigma &= {}_a\sigma({}_a^t\mathbf{r}, \theta) \\ \mathbf{m} &= \mathbf{m}({}_s^{\theta}\mathbf{J}^t, \theta) \end{aligned} \quad (129)$$

Furthermore, one also concludes that  ${}_s\sigma$  is not a function of  ${}_a^t\mathbf{r}$  and  ${}_s^{\theta}\mathbf{J}^t$ ,  ${}_a\sigma$  is not a function of  $\epsilon$  and  ${}_s^{\theta}\mathbf{J}^t$ , and  $\mathbf{m}$  is not a function of  $\epsilon$  and  ${}_a^t\mathbf{r}$ .

#### 4.1.8 Final form of conservation and balance laws

Equations resulting from the conservation and balance laws are summarized in the following.

$$\rho_0(\mathbf{x}) = \rho(\mathbf{x}, t) \quad (130)$$

$$\rho_0 \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b - \nabla \cdot {}_s\sigma - \nabla \cdot {}_a\sigma = 0 \quad (131)$$

$$\nabla \cdot \mathbf{m} - \epsilon : {}_a\sigma = 0 \quad (132)$$

$$\epsilon_{ijk} m_{ij} = 0 \quad (133)$$

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \mathbf{q} - \text{tr}([{}_s\sigma][\dot{\epsilon}]) - \text{tr}([{}_a\sigma][{}_a^t\dot{r}]) - \text{tr}([m][{}_s^{\theta}\dot{J}^t]) - {}_i\dot{\Theta} \cdot (\epsilon : \sigma) = 0 \quad (134)$$

$$\rho_0 \left( \frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr}([{}_s\sigma][\dot{\epsilon}]) - \text{tr}([{}_a\sigma][{}_a^t\dot{r}]) - \text{tr}([m][{}_s^{\theta}\dot{J}^t]) - {}_i\dot{\Theta} \cdot (\epsilon : \sigma) \leq 0 \quad (135)$$

In the mathematical model described by the above set of equations, the dependent variables are:  $\mathbf{v}$  (3),  ${}_s\sigma$  (6),  ${}_a\sigma$  (3),  $\mathbf{m}$  (6),  $\mathbf{q}$  (3),  $\theta$  (1),  ${}_e\Theta$  (3), a total of 25.  $\Phi$ ,  $e$ , and  $\eta$  are not dependent variables as these are expressed in terms of others. Numbers in brackets refer to the number of variables. The equations in the model are: linear momentum (3), angular momentum (3), energy (1), constitutive theories for  ${}_s\sigma$  (6),  ${}_a\sigma$  (3),  $\mathbf{m}$  (6),  $\mathbf{q}$  (3), a total of 25, hence the mathematical model has closure.



#### 4.1.9 Constitutive theories

Surana et al. [68,90,91] have presented ordered rate constitutive theories for non-classical solid continua incorporating internal and Cosserat rotation: thermoelastic solids, thermoviscoelastic solids without memory and thermoviscoelastic solids with memory. Details of the derivations of these theories are too many to include in this paper due to space consideration. Instead, we outline a clear procedure in the following based on references [68,90,91] that is rather straight forward to follow to derive the desired constitutive theories.

- (1) Generally entropy inequality suffices in the determination of the constitutive variables (in conjunction with principle of causality [67,69]), but other balance laws may also have to be examined in some cases. In general  $\Phi$ ,  $\eta$ ,  $\boldsymbol{\varepsilon}$ ,  $\mathbf{q}$  are constitutive variables for most simple continuous matter. Specific form of the choice of  $\boldsymbol{\sigma}$  i.e.  ${}_s\boldsymbol{\sigma}$ ,  ${}_a\boldsymbol{\sigma}$ ,  ${}_e({}_s\boldsymbol{\sigma})$ ,  ${}_d({}_s\boldsymbol{\sigma})$  depends upon the type of physics of solid matter under consideration. However, in this choice only symmetric and antisymmetric tensor are permitted (due to restriction by representation theorem). A non-symmetric tensor  $\boldsymbol{\sigma}$  is decomposed into  ${}_s\boldsymbol{\sigma}$  and  ${}_a\boldsymbol{\sigma}$ , symmetric and antisymmetric tensor. Furthermore inclusion of dissipation and memory requires that we decompose  ${}_s\boldsymbol{\sigma}$  into deviatoric and equilibrium tensors i.e.  ${}_d({}_s\boldsymbol{\sigma})$  and  ${}_e({}_s\boldsymbol{\sigma})$ . Similar concepts may also apply to Cauchy moment tensor as well, however we keep in mind that Cauchy moment tensor is symmetric. These decompositions are substituted in the entropy inequality to establish conjugate pairs. In each conjugate pair both tensors are either symmetric or antisymmetric.
- (2) Determination of the argument tensors of each constitutive variable are initiated using the conjugate pairs in the entropy inequality, these are augmented by including other tensor (if necessary) that are not obvious from the entropy inequality. Finally, based on principle of equipresence totality of all argument tensor appearing in each constitutive variable are considered as the argument tensors for the constitutive variables.
- (3) Now since we know the argument tensors of  $\Phi$ , material derivative of  $\Phi$ , i.e.  $\frac{D\Phi}{Dt}$  is established using chain rule of differentiation and then substituted in the entropy inequality. The terms in the entropy inequality are regrouped and conditions are established under which entropy inequality will be satisfied. This process almost always eliminate  $\eta$  as a constitutive variable and shortens the list of argument tensors of  $\Phi$  appreciably.
- (4) At this state matter specific physics need to be considered to precisely define the choice of stress and moment tensors as constitutive variables, as well as their argument tensors. We consider some details in the following.
  - (a) Thermoelastic solids [68]
 

In this case decomposition  $\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}$  is needed and we obtain the following from step (3)

$$\Phi = \Phi(\boldsymbol{\varepsilon}_s^\theta, \mathbf{J}^t, \theta) \quad (136)$$

Entropy inequality

$$\eta = -\frac{\partial \Phi}{\partial \theta} \quad (137)$$

$${}_s\sigma_{ki} = \rho_0 \frac{\partial \Phi}{\partial \epsilon_{ki}}; \Phi = \Phi(I_\epsilon, II_\epsilon, III_\epsilon, \theta) \quad (138)$$

or

$${}_s\sigma = {}_s\sigma(\epsilon, \theta) \quad (139)$$

$${}_a\sigma = {}_a\sigma({}_a^t r, \theta) \quad (140)$$

$$m_{ki} = \rho_0 \frac{\partial \Phi}{\partial ({}_s^{\theta} J^t)_{ki}}; \Phi = \Phi(I({}_s^{\theta} J^t), II({}_s^{\theta} J^t), III({}_s^{\theta} J^t), \theta) \quad (141)$$

or

$$m = m({}_s^{\theta} J^t, \theta) \quad (142)$$

$$\frac{q \cdot g}{\theta} \leq 0 \quad (143)$$

$$q = q(g, \theta) \quad (144)$$

$${}_i \dot{\theta} \cdot (\epsilon : \sigma) = 0 \quad (145)$$

and

$$\text{tr}([{}_s\sigma][\dot{\epsilon}]) > 0, \text{tr}([m][{}_s^{\theta} J^t]) > 0, \text{tr}([{}_a\sigma][{}_a^t \dot{r}]) > 0$$

Constitutive theories for  ${}_s\sigma$  and  $m$  can be derived using (138) and (141) or alternatively using (139) and (142) and representation theorem [73–89]. The final forms resulting from these two approaches are same. The material coefficients are established by expanding the unknown coefficients in the constitutive theories in Taylor series in the combined invariants of the argument tensor and temperature  $\theta$  [67,73–89].

(b) Thermoviscoelastic solids without memory [90]

$$\sigma = {}_s\sigma + {}_a\sigma; {}_s\sigma = {}_e({}_s\sigma) + {}_d({}_s\sigma)$$

Complete derivation of the constitutive theories is given by Surana et al. [90]. We summarize main steps in the following.

For incompressible solid (small deformation, small strain) we have

$${}_e({}_s\sigma) = p(\theta)I \quad (146)$$

Entropy inequality reduces to

$$- \text{tr}([{}_d({}_s\sigma)][\dot{\epsilon}_{[0]}]) - \text{tr}([{}_a\sigma][{}_d^t\dot{r}_{[0]}]) - \text{tr}([m][{}_s^{\theta}j_{[0]}^t]) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - {}_i\dot{\Theta} \cdot (\boldsymbol{\epsilon} : \sigma) \leq 0 \quad (147)$$

The entropy inequality in (147) is satisfied if

$$\begin{aligned} {}_s\sigma\psi &= \text{tr}([{}_d({}_s\sigma)][\dot{\epsilon}_{[0]}]) \geq 0 \\ {}_a\sigma\psi &= \text{tr}([{}_a\sigma][{}_d^t\dot{r}_{[0]}]) \geq 0 \\ {}_m\psi &= \text{tr}([m][{}_s^{\theta}j_{[0]}^t]) \geq 0 \end{aligned} \quad (148)$$

$$\frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \quad (149)$$

$${}_i\dot{\Theta} \cdot (\boldsymbol{\epsilon} : \sigma) = 0 \quad (\text{constraint equation}) \quad (150)$$

Inequalities (148) imply that the rate of work should be positive, or zero if the physics causing these is absent. Equation (150) is a constraint equation that is necessary to ensure the second law of thermodynamics is satisfied. Consideration of ordered rate constitutive theories [90] require that we consider

$$\begin{aligned} {}_d({}_s\sigma) &= {}_d({}_s\sigma)(\boldsymbol{\epsilon}_{[i]}; i = 0, 1, \dots, n_{\epsilon}, \theta) \\ \mathbf{m} &= \mathbf{m}({}_s^{\theta}j_{[i]}^t; i = 0, 1, \dots, n_{({}_s^{\theta}j)}, \theta) \\ {}_a\sigma &= {}_a\sigma({}_d^t r_{[i]}; i = 0, 1, \dots, n_{({}_d^t r)}, \theta) \\ \mathbf{q} &= \mathbf{q}(\mathbf{g}, \theta) \end{aligned} \quad (151)$$

Constitutive theories for  ${}_d({}_s\sigma)$ ,  $\mathbf{m}$ ,  ${}_a\sigma$  and  $\mathbf{q}$  are derived using representation theorem and the material coefficients are established using Taylor series expansion of the coefficients in the linear combination of the generators in the constitutive theories. See Surana et al. [90] for more details.

(c) Thermoviscoelastic solids with memory [91]

In this case also we have

$$\sigma = {}_s\sigma + {}_a\sigma; {}_s\sigma = {}_e({}_s\sigma) + {}_d({}_s\sigma)$$

and

$${}_e({}_s\sigma) = p(\theta)\mathbf{I} \quad (152)$$

the entropy inequality reduces to

$$- \text{tr}([{}_d({}_s\sigma^{[0]})][\dot{\epsilon}_{[0]}]) - \text{tr}([{}_a\sigma^{[0]}][{}_d^t\dot{r}_{[0]}]) - \text{tr}([m_{[0]}][{}_s^{\theta}j_{[0]}^t]) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - {}_i\dot{\Theta} \cdot (\boldsymbol{\epsilon} : \sigma) \leq 0 \quad (153)$$

superscript [0] and subscript [0] refer to material derivatives of order zero for contra and co-variant measures which are same for small deformation strain.

The entropy inequality in (153) is satisfied if

$$\begin{aligned} {}_s\sigma\psi &= \text{tr}([{}_d({}_s\sigma^{[0]})][\dot{\epsilon}_{[0]}]) \geq 0 \\ {}_a\sigma\psi &= \text{tr}([{}_a\sigma^{[0]}][{}_d^t\dot{r}_{[0]}]) \geq 0 \\ {}_m\psi &= \text{tr}([m_{[0]}][{}_s^{\theta}j_{[0]}^t]) \geq 0 \end{aligned} \quad (154)$$

$$\frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \quad (155)$$

$$_i \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) = 0 \quad (\text{constraint equation}) \quad (156)$$

Inequalities (154) imply that the rate of work should be positive, or zero if the physics causing these is absent. Equation (156) is a constraint equation that is necessary to ensure that the second law of thermodynamics is satisfied.

The final set of constitutive variables and their argument tensors are, keeping in mind that  ${}_s \boldsymbol{\sigma}^{[m_{s\sigma}]}$  needs to be replaced by  ${}_d({}_s \boldsymbol{\sigma}^{[m_{s\sigma}]})$  and its argument tensors  ${}_s \boldsymbol{\sigma}^{[j]}$ ;  $j = 0, 1, \dots, (m_{s\sigma} - 1)$  must be replaced by  ${}_d({}_s \boldsymbol{\sigma}^{[j]})$ ;  $j = 0, 1, \dots, (m_{s\sigma} - 1)$  and noting that  $\boldsymbol{\epsilon}_{[0]}$  is no longer an argument tensor of  $\Phi$  are given by [91].

$$\begin{aligned} \Phi &= \Phi(\theta) \\ {}_d({}_s \boldsymbol{\sigma}^{[m_{s\sigma}]}) &= {}_d({}_s \boldsymbol{\sigma}^{[m_{s\sigma}]})({}_s \boldsymbol{\epsilon}_{[i]}; i = 0, 1, \dots, n_{\epsilon}, {}_d({}_s \boldsymbol{\sigma}^{[j]}); j = 0, 1, \dots, (m_{s\sigma} - 1), \theta) \\ {}_a \boldsymbol{\sigma}^{[m_{a\sigma}]} &= {}_a \boldsymbol{\sigma}^{[m_{a\sigma}]}({}_a \mathbf{r}_{[i]}; i = 0, 1, \dots, n_{(a)r}, {}_a \boldsymbol{\sigma}^{[j]}; j = 0, 1, \dots, (m_{a\sigma} - 1), \theta) \\ \mathbf{m}^{[m_m]} &= \mathbf{m}^{[m_m]}({}_s \mathbf{J}_{[i]}^t; i = 0, 1, \dots, n_{(s)J^t}, \mathbf{m}^{[j]}; j = 0, 1, \dots, (m_m - 1), \theta) \\ \mathbf{q} &= \mathbf{q}(\mathbf{g}, \theta) \end{aligned} \quad (157)$$

Constitutive theories are derived using representation theorem and the material coefficients are established using Taylor series expansion of the coefficients in the constitutive theory used to establish linear combination of the generators. See Surana et al. [91] for details as well as for the derivation of memory modulus and the specialized form of the general constitutive theory for the constitutive models similar to Maxwell, Oldroyd-B and Giesekus.

#### 4.2 Conservation and balance laws and constitutive theories for solid continua incorporating internal rotations only

The conservation and the balance laws for the non-classical continuum theory incorporating internal rotations only [92,93] can be easily deduced from (130) – (135) by letting

$$[{}_a^e \gamma] = 0; {}_t \Theta = {}_i \Theta; [{}_a^t r] = [{}_a^i r]; [{}^\Theta J^t] = [{}^\Theta J^i] = \left[ \frac{\partial \{ {}_i \Theta \}}{\partial \{ x \}} \right] \quad (158)$$

and noting that

$$\text{tr}([{}_a \sigma][{}_a^t \dot{r}]) = \text{tr}([{}_a \sigma][{}_a^i \dot{r}]) = -{}_i \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma})$$

Thus, (130) – (135) reduces to

$$\begin{aligned} \rho_0(\mathbf{x}) &= \rho(\mathbf{x}, t) \\ \rho_0 \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b - \nabla \cdot {}_s \boldsymbol{\sigma} - \nabla \cdot {}_a \boldsymbol{\sigma} &= 0 \\ \nabla \cdot \mathbf{m} - \boldsymbol{\epsilon} : {}_a \boldsymbol{\sigma} &= 0 \\ \epsilon_{ijk} m_{ij} &= 0 \\ \rho_0 \frac{D\epsilon}{Dt} + \nabla \cdot \mathbf{q} - \text{tr}([{}_s \sigma][\dot{\epsilon}]) - \text{tr}([m][{}_s^{\Theta} \dot{J}^i]) &= 0 \\ \rho_0 \left( \frac{D\phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr}([{}_s \sigma][\dot{\epsilon}]) - \text{tr}([m][{}_s^{\Theta} \dot{J}^i]) &\leq 0 \end{aligned} \quad (159)$$

The same mathematical model has also been derived by Surana et al. [92,93] by considering rotations  ${}^i\boldsymbol{\theta}$  only.

In the mathematical model described by the above set of equations, the dependent variables are:  $\mathbf{v}$  (3),  ${}_s\boldsymbol{\sigma}$  (6),  ${}_a\boldsymbol{\sigma}$  (3),  $\mathbf{m}$  (6),  $\mathbf{q}$  (3),  $\theta$  (1), a total of 22,  $\Phi$ ,  $e$ , and  $\eta$  are not dependent variables as these are expressed in terms of others. Numbers in brackets refer to the number of variables. The equations in the model are: linear momentum (3), angular momentum (3), energy (1), constitutive theories for  ${}_s\boldsymbol{\sigma}$  (6),  $\mathbf{m}$  (6),  $\mathbf{q}$  (3), a total of 22, hence the mathematical model has closure.

We note that in this case  ${}_a\boldsymbol{\sigma}$  are not constitutive variables. Constitutive theories for  ${}_s\boldsymbol{\sigma}$ ,  $\mathbf{m}$  and  $\mathbf{q}$  are established following the same procedure as described in section 4.1.9. Some details are given in the following for thermoelastic solids and thermoviscoelastic solids with and without memory.

a) Thermoelastic solid [94]

The entropy inequality reduces to

$$\begin{aligned} & \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr}([{}_s\boldsymbol{\sigma}][\dot{\boldsymbol{\varepsilon}}]) - \text{tr}([\mathbf{m}][{}_s^{\theta}\dot{\mathbf{j}}^i]) \leq 0 \\ \text{and} \\ & {}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \\ \text{or} \\ & {}_s\sigma_{ki} = \rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}}(I_{\varepsilon}, II_{\varepsilon}, III_{\varepsilon}, \theta) \\ & \mathbf{m} = \mathbf{m}({}_s^{\theta}\mathbf{j}^i, \theta) \\ \text{or} \\ & m_{ki} = \rho_0 \frac{\partial \Phi}{\partial ({}_s^{\theta}j_{ki}^i)}(I_{({}_s^{\theta}\mathbf{j}^i)}, II_{({}_s^{\theta}\mathbf{j}^i)}, III_{({}_s^{\theta}\mathbf{j}^i)}, \theta) \\ & \mathbf{q} = \mathbf{q}(\mathbf{g}, \theta) \end{aligned} \quad (160)$$

b) Thermoviscoelastic solid without memory [95]

$${}_s\boldsymbol{\sigma} = {}_e({}_s\boldsymbol{\sigma}) + {}_d({}_s\boldsymbol{\sigma})$$

The entropy inequality reduces to

$$\begin{aligned} & \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \text{tr}([{}_d({}_s\boldsymbol{\sigma})][\dot{\boldsymbol{\varepsilon}}_{[0]}]) - \text{tr}([\mathbf{m}][{}_s^{\theta}\dot{\mathbf{j}}^i]) \leq 0 \\ & {}_e({}_s\boldsymbol{\sigma}) = p(\theta)\mathbf{I}; \quad \text{incompressible} \\ & {}_d({}_s\boldsymbol{\sigma}) = {}_d({}_s\boldsymbol{\sigma})(\boldsymbol{\varepsilon}_{[i]}; i = 0, 1, \dots, n_{\varepsilon}, \theta) \\ & \mathbf{m} = \mathbf{m}({}_s^{\theta}\mathbf{j}_{[i]}^i; i = 0, 1, \dots, n_{({}_s^{\theta}\mathbf{j}^i)}, \theta) \\ & \mathbf{q} = \mathbf{q}(\mathbf{g}, \theta) \end{aligned} \quad (161)$$

and

$$\frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0; \quad \text{tr}([{}_d({}_s\boldsymbol{\sigma})][\dot{\boldsymbol{\varepsilon}}_{[0]}]) > 0 \quad \text{and} \quad \text{tr}([\mathbf{m}][{}_s^{\theta}\dot{\mathbf{j}}_{[0]}^i]) > 0$$

c) Thermoviscoelastic solid with memory [96]

Here also we have

$$\begin{aligned} & {}_s\boldsymbol{\sigma} = {}_e\boldsymbol{\sigma} + {}_d({}_s\boldsymbol{\sigma}) \\ & {}_e({}_s\boldsymbol{\sigma}) = p(\theta)\mathbf{I}; \quad \text{incompressible} \end{aligned} \quad (162)$$

The entropy inequality reduces to

$$-\operatorname{tr}([{}_d({}_s\sigma^{[0]})][\dot{\epsilon}_{[0]}]) - \operatorname{tr}([m^{[0]}]{}_s^{\theta}j^{[i]}) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \quad (163)$$

The entropy inequality is satisfied if

$$\begin{aligned} {}_s\sigma\psi &= \operatorname{tr}([{}_d({}_s\sigma^{[0]})][\dot{\epsilon}_{[0]}]) \geq 0 \\ {}^m\psi &= \operatorname{tr}([m^{[0]}]{}_s^{\theta}j^{[i]}) \geq 0 \\ \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} &\leq 0 \end{aligned} \quad (164)$$

Condition (164) imply that rate of work due to  ${}_s\sigma^{[0]}$  and  $m^{[0]}$  must be positive. The final set of constitutive variables and their argument tensors are, keeping in mind that  ${}_s\sigma^{[m_s\sigma]}$  and its argument tensors  ${}_s\sigma^{[j]}$ ;  $j = 0, 1, \dots, (m_s\sigma - 1)$  need to be replaced with corresponding deviatoric stress tensor and its rates.

$$\begin{aligned} \Phi &= \Phi(\theta) \\ d({}_s\sigma^{[m_s\sigma]}) &= d({}_s\sigma^{[m_s\sigma]})(\epsilon_{[j]}; i = 0, 1, \dots, n_\epsilon, d({}_s\sigma^{[j]}); j = 0, 1, \dots, (m_s\sigma - 1), \theta) \\ \mathbf{m}^{[m_m]} &= \mathbf{m}^{[m_m]}({}_s^{\theta}j^{[i]}; i = 0, 1, \dots, n({}_s^{\theta}j), \mathbf{m}^{[j]}; j = 0, 1, \dots, (m_m - 1), \theta) \\ \mathbf{q} &= \mathbf{q}(\mathbf{g}, \theta) \end{aligned} \quad (165)$$

Constitutive theory and the material coefficients are established using representation theorem and the Taylor series expansion about a known configuration.

#### 4.3 Conservation and balance laws for fluent continua incorporating internal and Cosserat rotation rates

This work was first reported by Surana et al. [97]. In the derivation of the conservation and the balance laws for the fluent continua the choice of basis for Cauchy stress tensor is important. We can choose Cauchy stress tensor  $\bar{\sigma}^{(0)}$  or  $\bar{\sigma}_{(0)}$  based on contra-and-covariant measures. This choice establishes the choice of the conjugate strain rate measure.  $\bar{\sigma}^{(0)}$  requires that we choose first convected time derivative of Green's strain tensor ( $\mathbf{Y}_{(1)}$ ) as conjugate to  $\bar{\sigma}^{(0)}$ . Likewise  $\mathbf{Y}^{(1)}$ , a contravariant strain rate measure, the convected time derivative of Almansi strain tensor is conjugate to covariant Cauchy stress tensor  $\bar{\sigma}_{(0)}$ . In general  $\bar{\sigma}^{(i)}$ ;  $i = 0, 1, \dots, \underline{m}$  the convected time derivatives of orders 0, 1, ..., etc. are conjugate to  $[\mathbf{Y}_{(j)}]$ ;  $j = 1, 2, \dots, \underline{n}$  convected time derivatives of the Green's strain tensor of orders 1, 2, ...,  $\underline{n}$ . Likewise  $\bar{\sigma}_{(i)}$ ;  $i = 0, 1, \dots, \underline{m}$  are conjugate to  $[\mathbf{Y}^{(j)}]$ ;  $j = 1, 2, \dots, \underline{n}$ , the convected time derivatives of orders 1, 2, ...,  $\underline{n}$  of the Almansi strain tensor. In the derivation of the conservation and balance laws we choose  $\bar{\sigma}^{(i)}$ ;  $i = 0, 1, \dots, \underline{m}$  and  $[\mathbf{Y}_{(j)}]$ ;  $j = 1, 2, \dots, \underline{n}$  as conjugate pairs. We recall that  $[\mathbf{Y}^{(1)}] = [\mathbf{Y}_{(1)}] = [D]$ , the symmetric part of the velocity gradient tensor.

#### 4.3.1 Conservation of mass

Conservation of mass in a deforming volume of fluid leads to continuity equation that remains the same in the present work as for classical continuum theory [67,69,70] and is given in the Eulerian description for compressible fluent continua.

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\nabla} \cdot (\bar{\rho} \bar{\mathbf{v}}) = 0 \quad (166)$$

or

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho} \operatorname{div}(\bar{\mathbf{v}}) = 0 \quad (167)$$

in which  $\bar{\rho}(\bar{\mathbf{x}}, t)$  is the density at a material point at  $\bar{\mathbf{x}}$  in the current configuration.

#### 4.3.2 Balance of linear momenta

For a deforming volume of matter, the rate of change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation is exactly same as in case of classical continuum mechanics. Following reference [67] and using contravariant Cauchy stress tensor  $\bar{\boldsymbol{\sigma}}^{(0)}$ , we can write the following.

$$\bar{\rho} \frac{D\bar{\mathbf{v}}}{Dt} - \bar{\rho} \bar{\mathbf{F}}^b - \bar{\nabla} \cdot \bar{\boldsymbol{\sigma}}^{(0)} = 0 \quad (168)$$

or

$$\bar{\rho} \frac{\partial \bar{v}_i}{\partial t} + \bar{\rho} \bar{v}_j \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho} \bar{F}_i^b - \frac{\partial \bar{\sigma}_{ji}^{(0)}}{\partial \bar{x}_j} = 0 \quad (169)$$

in which  $\bar{\mathbf{F}}^b$  are body forces per unit mass and  $\bar{\boldsymbol{\sigma}}^{(0)}$  is contravariant Cauchy stress tensor (see reference [67] for using covariant Cauchy stress tensor  $\bar{\boldsymbol{\sigma}}_{(0)}$  or Jaumann stress tensor  ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$  in place of  $\bar{\boldsymbol{\sigma}}^{(0)}$  and the consequences of doing so). Equations (168) or (169) are called momentum equations in  $x_1$ ,  $x_2$ , and  $x_3$  directions.

#### 4.3.3 Balance of angular momenta

The principle of balance of angular momenta for a non-classical continuum can be stated as: *The material derivative (time rate of change) of moments of momenta must be equal to the vector sum of the moments of forces and the moments.* Thus, due to the surface stress  $\bar{\mathbf{P}}$ , total surface moment  $\bar{\mathbf{M}}$  (per unit area), body force  $\bar{\mathbf{F}}^b$  (per unit mass), and the momentum  $\bar{\rho} \bar{\mathbf{v}} d\bar{V}$  for an elemental mass  $\bar{\rho} d\bar{V}$  in the current configuration we can write the following in Eulerian description.

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{v}} d\bar{V} = \int_{\partial \bar{V}(t)} (\bar{\mathbf{x}} \times \bar{\mathbf{P}} - \bar{\mathbf{M}}) d\bar{A} + \int_{\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{F}}^b d\bar{V} \quad (170)$$

Negative sign for  $\bar{\mathbf{M}}$  is due to clockwise rotation rates being positive. We consider each term in (170) individually. First consider

$$\begin{aligned}
\frac{D}{Dt} \int_{V(t)} \bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{v}} d\bar{V} &= \frac{D}{Dt} \int_{V(t)} \epsilon_{ijk} \bar{x}_i \bar{v}_j \mathbf{e}_k \bar{\rho} d\bar{V} \\
&= \frac{D}{Dt} \int_V \epsilon_{ijk} x_i v_j \mathbf{e}_k \rho_0 dV \\
&= \int_V \epsilon_{ijk} \mathbf{e}_k \frac{D}{Dt} (x_i v_j) \rho_0 dV \\
&= \int_{\bar{V}(t)} \epsilon_{ijk} \mathbf{e}_k \frac{D}{Dt} (\bar{x}_i \bar{v}_j) \bar{\rho} d\bar{V} \\
&= \int_{\bar{V}(t)} \epsilon_{ijk} \mathbf{e}_k \left( \frac{D\bar{x}_i}{Dt} \bar{v}_j + \bar{x}_i \frac{D\bar{v}_j}{Dt} \right) \bar{\rho} d\bar{V} \\
&= \int_{\bar{V}(t)} \epsilon_{ijk} \mathbf{e}_k \left( \bar{v}_i \bar{v}_j + \bar{x}_i \frac{D\bar{v}_j}{Dt} \right) \bar{\rho} d\bar{V} \\
&= \int_{\bar{V}(t)} \epsilon_{ijk} \mathbf{e}_k \left( \bar{x}_i \frac{D\bar{v}_j}{Dt} \right) \bar{\rho} d\bar{V} ; \quad \epsilon_{ij} \bar{v}_i \bar{v}_j = 0
\end{aligned} \tag{171}$$

Consider the first term on the right hand side of (170). Using contravariant Cauchy moment tensor  $\bar{\mathbf{m}}^{(0)}$  we can write

$$\begin{aligned}
\int_{\partial\bar{V}(t)} (\bar{\mathbf{x}} \times \bar{\mathbf{P}} - \bar{\mathbf{M}}) d\bar{A} &= \int_{\partial\bar{V}(t)} \left( \bar{\mathbf{x}} \times (\bar{\boldsymbol{\sigma}}^{(0)})^T \cdot \bar{\mathbf{n}} - (\bar{\mathbf{m}}^{(0)})^T \cdot \bar{\mathbf{n}} \right) d\bar{A} \\
&= \int_{\partial\bar{V}(t)} \mathbf{e}_k \left( \epsilon_{ijk} \bar{x}_i \bar{\sigma}_{mj}^{(0)} \bar{n}_m - \bar{m}_{mk}^{(0)} \bar{n}_m \right) d\bar{A} \\
&= \int_{\bar{V}(t)} \mathbf{e}_k \left( \epsilon_{ijk} (\bar{x}_i \bar{\sigma}_{mj}^{(0)})_{,m} - \bar{m}_{mk,m}^{(0)} \right) d\bar{V} \\
&= \int_{\bar{V}(t)} \mathbf{e}_k \left( \epsilon_{ijk} (\bar{x}_{i,m} \bar{\sigma}_{mj}^{(0)} + \bar{x}_i \bar{\sigma}_{mj,m}^{(0)}) - \bar{m}_{mk,m}^{(0)} \right) d\bar{V} \\
&= \int_{\bar{V}(t)} \mathbf{e}_k \left( \epsilon_{ijk} (\bar{\sigma}_{ij}^{(0)} + \bar{x}_i \bar{\sigma}_{mj,m}^{(0)}) - \bar{m}_{mk,m}^{(0)} \right) d\bar{V}
\end{aligned} \tag{172}$$

Next consider the second term on the right hand side (170).

$$\int_{\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\rho} \bar{\mathbf{F}}^b d\bar{V} = \int_{\bar{V}(t)} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \bar{F}_j^b \bar{\rho} d\bar{V} \tag{173}$$

Substituting from (171), (172), and (173) into (170)

$$\int_{\bar{V}(t)} \mathbf{e}_k \epsilon_{ijk} \left( \bar{x}_i \frac{D\bar{v}_j}{Dt} \right) \bar{\rho} d\bar{V} = \int_{\bar{V}(t)} \mathbf{e}_k \left( \epsilon_{ijk} (\bar{\sigma}_{ij}^{(0)} + \bar{x}_i \bar{\sigma}_{mj,m}^{(0)}) - \bar{m}_{mk,m}^{(0)} \right) d\bar{V} + \int_{\bar{V}(t)} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \bar{F}_j^b \bar{\rho} d\bar{V} \tag{174}$$

or



$$\int_{\bar{V}(t)} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \left( \bar{\rho} \frac{D\bar{v}_j}{Dt} - \bar{\rho} \bar{F}_j^b - \bar{\sigma}_{mk,m}^{(0)} \right) d\bar{V} + \int_{\bar{V}(t)} \mathbf{e}_k \left( \bar{m}_{mk,m}^{(0)} - \epsilon_{ijk} \bar{\sigma}_{ij}^{(0)} \right) d\bar{V} = 0 \quad (175)$$

Using balance of linear momenta (169), (175) reduces to

$$\int_{\bar{V}(t)} \mathbf{e}_k \left( \bar{m}_{mk,m}^{(0)} - \epsilon_{ijk} \bar{\sigma}_{ij}^{(0)} \right) d\bar{V} = 0 \quad (176)$$

Since volume  $\bar{V}$  is arbitrary, we have

$$\bar{m}_{mk,m}^{(0)} - \epsilon_{ijk} \bar{\sigma}_{ij}^{(0)} = 0 \quad (177)$$

or

$$\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)} - \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = \mathbf{0} \quad (178)$$

Equation (178) represents balance of angular momenta. The contravariant Cauchy stress tensor  $\bar{\boldsymbol{\sigma}}^{(0)}$  is non-symmetric and so is the contravariant Cauchy moment tensor  $\bar{\mathbf{m}}^{(0)}$ .

### Remarks

- From the balance of angular momenta (178), we note that the antisymmetric components of the Cauchy stress tensor are balanced by the gradients of the Cauchy moment tensor (non-symmetric at this stage).
- In case of classical continuum theory for fluent continua, the balance of angular momenta is a statement of self-equilibrating moments due to symmetry of Cauchy stress tensor.

$$\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = 0 \quad (179)$$

- In case of non-classical fluent continua, the existence of the Cauchy moment tensor  $\bar{\mathbf{m}}^{(0)}$  due to material constitution resisting the rotation rates, both internal and Cosserat, results in shear stress couples from the antisymmetric part of the Cauchy stress tensor that are balanced by internal moments. Thus, for non-classical continuum theory balance of angular momenta yields (178) instead of (179) that only holds in case of classical continuum theory.
- It is important to point out that the theory presented here does not assume the existence of Cauchy moment  $\bar{\mathbf{m}}^{(0)}$ . The non-classical continuum theory presented in the paper demonstrates that varying rotation rates, both internal and Cosserat, at neighboring material points when resisted by the deforming fluent continua necessitate existence of the Cauchy moment tensor. The balance of angular momenta establishes a relationship between  $\bar{\mathbf{m}}^{(0)}$  and  $\bar{\boldsymbol{\sigma}}^{(0)}$ .

#### 4.3.4 Balance of moment of moments balance law

The rationale for this balance law is similar to what has been described for solid continua, except that in case of fluent continua we have rotation rates instead of rotation. Thus, the material in [section 4.1.4](#) for solid continua demonstrating the need for this balance law is

applicable here as well, hence not repeated. In the following we do present the derivation [72] as the Cauchy moment measure in this case is basis dependent.

In this balance law we must consider the rate of moment of angular momenta due to rotation rates to balance with the moment of the moments of the antisymmetric components of the Cauchy stress tensor and the moment of  $\bar{\mathbf{M}}$ , all of which are only related to the non-classical physics due to rotation rates and their conjugate moments. We can write

$$\left( \begin{array}{c} \text{Rate of the moment} \\ \text{of the angular momenta} \\ \text{due to internal rotation} \\ \text{rates over volume } \bar{V}(t) \end{array} \right) = \left( \begin{array}{c} \text{Moment of moments} \\ \text{due to antisymmetric} \\ \text{components of Cauchy} \\ \text{stress tensor over } \bar{V}(t) \end{array} \right) - \left( \begin{array}{c} \text{Moment of} \\ \text{moment } \bar{\mathbf{M}} \\ \text{over } \partial\bar{V}(t) \end{array} \right) \quad (180)$$

Negative sign is due to the assumption of clockwise rotation rates and the corresponding moment tensor to be positive. *We note that in continuum theories for continuous media we assume that the material points or particles have mass but no dimensions, thus the angular momenta due to rotation rates are zero at each material point as a material point can not have rotational inertial effects.* Thus, (180) reduces to

$$\left( \begin{array}{c} \text{Moment of moments} \\ \text{due to antisymmetric} \\ \text{components of Cauchy} \\ \text{stress tensor over } \bar{V}(t) \end{array} \right) - \left( \begin{array}{c} \text{Moment of} \\ \text{moment } \bar{\mathbf{M}} \\ \text{over } \partial\bar{V}(t) \end{array} \right) = 0 \quad (181)$$

If we consider current configuration at time  $t$ , then in Eulerian description, we can write (181) as

$$\int_{\bar{V}(t)} \bar{\mathbf{x}} \times \left( \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} \right) d\bar{V} - \int_{\partial\bar{V}(t)} \bar{\mathbf{x}} \times \bar{\mathbf{M}} \, d\bar{A} = 0 \quad (182)$$

### Remarks

- (1) In the absence of (180), a rate statement, if we consider (181) and (182) directly, then we could mistakenly view (181) and (182) as equilibrium of moment of moments, a static consideration. This is obviously incorrect. As stated by Yang et al. [66], (182) indeed is a balance law, even though its derivation based on a statement like (180) is not presented in [66]. Due to the fact that left side of (180) is zero, this balance law, (180), results into (182) which has the appearance of an equilibrium law.
- (2) Henceforth in this paper we refer to (182) as a *balance of moment of moments balance law*. For the non-classical physics considered in this paper the balance law (180) reduces to (182) which can also be labeled as equilibrium of moment of moments due to the absence of rate term (as it is zero). Thus, we can also say *balance/equilibrium of moment of moments law*.

We expand the second term in (182) and then convert the integral over  $\partial\bar{V}$  to the integral over  $\bar{V}$  using the divergence theorem:

$$\begin{aligned}
\int_{\partial \bar{V}} \bar{\mathbf{x}} \times \bar{\mathbf{M}} \, d\bar{A} &= \int_{\partial \bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \bar{M}_j d\bar{A} \\
&= \int_{\partial \bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{x}_i \bar{m}_{mj}^{(0)} \bar{n}_m d\bar{A} \\
&= \int_{\bar{V}} \mathbf{e}_k \left( \epsilon_{ijk} \bar{x}_i \bar{m}_{mj}^{(0)} \right)_{,m} d\bar{V} \\
&= \int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} \left( \bar{m}_{ij}^{(0)} + \bar{x}_i \bar{m}_{mj,m}^{(0)} \right) d\bar{V} \\
&= \int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{m}_{ij}^{(0)} d\bar{V} + \int_{\bar{V}} \bar{\mathbf{x}} \times (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) d\bar{V}
\end{aligned} \tag{183}$$

Using Equation (183) in (182) and collecting terms

$$\int_{\bar{V}} \bar{\mathbf{x}} \times \left( -\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} \right) d\bar{V} - \int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{m}_{ij}^{(0)} d\bar{V} = 0 \tag{184}$$

The first term in (184) vanishes due to balance of angular momenta, giving the condition

$$\int_{\bar{V}} \mathbf{e}_k \epsilon_{ijk} \bar{m}_{ij}^{(0)} d\bar{V} = 0 \tag{185}$$

which, because  $\bar{V}$  is arbitrary, yields

$$\epsilon_{ijk} \bar{m}_{ij}^{(0)} = 0 \tag{186}$$

Equation (186) implies that Cauchy moment tensor  $\bar{\mathbf{m}}^{(0)}$  is symmetric in non-classical continuum theories when balance of moment of moments is used as an additional balance law.

### Remarks

- (1) We note that  $\bar{\mathbf{m}}^{(0)}$  in balance of angular momenta (178) was nonsymmetric, but now is symmetric due to consideration of balance of moment of moments as a balance law.
- (2) In the energy equation and in the entropy inequality  $\bar{\mathbf{m}}^{(0)}$  will be symmetric as well.

#### 4.3.5 First law of thermodynamics

The sum of work and heat added to a deforming volume of matter must result in increase of the energy of the system [67]. This is expressed as a rate equation in Eulerian description in the following.

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{187}$$

$\bar{E}_t$ ,  $\bar{Q}$ , and  $\bar{W}$  are total energy, heat added, and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt} \int_{V(t)} \bar{\rho} \left( \bar{e} + \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} - \bar{\mathbf{F}}^b \cdot \bar{\mathbf{u}} \right) d\bar{V} \quad (188)$$

$$\frac{D\bar{Q}}{Dt} = - \int_{\partial V(t)} \bar{\mathbf{q}} \cdot \bar{\mathbf{n}} d\bar{A} \quad (189)$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial V(t)} (\bar{\mathbf{P}} \cdot \bar{\mathbf{v}} + \bar{\mathbf{M}} \cdot {}^t\bar{\boldsymbol{\Theta}}) d\bar{A} \quad (190)$$

where  $\bar{e}$  is specific internal energy,  $\bar{\mathbf{F}}^b$  is body force vector per unit mass, and  $\bar{\mathbf{q}}$  is rate of heat. Note that the additional term  $\bar{\mathbf{M}} \cdot {}^t\bar{\boldsymbol{\Theta}}$  in  $\frac{D\bar{W}}{Dt}$  contributes additional rate of work due to rates of total rotations  ${}^t\bar{\boldsymbol{\Theta}}$ . Expanding integrals and following reference [67], one can show the following.

$$\begin{aligned} \frac{D}{Dt} \int_{V(t)} \bar{\rho} \left( \bar{e} + \frac{1}{2} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} - \bar{\mathbf{F}}^b \cdot \bar{\mathbf{u}} \right) d\bar{V} &= \int_V \left( \rho_0 \frac{De}{Dt} + \rho_0 \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} - \rho_0 \mathbf{F}^b \cdot \mathbf{v} \right) dV \\ &= \int_{V(t)} \left( \bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\rho} \bar{\mathbf{v}} \cdot \frac{D\bar{\mathbf{v}}}{Dt} - \bar{\rho} \bar{\mathbf{F}}^b \cdot \bar{\mathbf{v}} \right) d\bar{V} \end{aligned} \quad (191)$$

$$- \int_{\partial V(t)} \bar{\mathbf{q}} \cdot \bar{\mathbf{n}} d\bar{A} = - \int_{V(t)} \bar{\nabla} \cdot \bar{\mathbf{q}} d\bar{V} \quad (192)$$

$$\bar{\mathbf{P}} \cdot \bar{\mathbf{v}} d\bar{A} = \bar{\mathbf{v}} \cdot (\bar{\boldsymbol{\sigma}}^{(0)})^T \cdot \bar{\mathbf{n}} \quad d\bar{A} = (\bar{\mathbf{v}} \cdot (\bar{\boldsymbol{\sigma}}^{(0)})^T) \cdot d\bar{\mathbf{A}} \quad (193)$$

$$\bar{\mathbf{M}} \cdot {}^t\bar{\boldsymbol{\Theta}} d\bar{A} = {}^t\bar{\boldsymbol{\Theta}} \cdot (\bar{\mathbf{m}}^{(0)})^T \cdot \bar{\mathbf{n}} \quad d\bar{A} = ({}^t\bar{\boldsymbol{\Theta}} \cdot (\bar{\mathbf{m}}^{(0)})^T) \cdot d\bar{\mathbf{A}} \quad (194)$$

Thus, we can write the following for (187).

$$\begin{aligned} \int_{V(t)} \left( \bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\rho} \bar{\mathbf{v}} \cdot \frac{D\bar{\mathbf{v}}}{Dt} - \bar{\rho} \bar{\mathbf{F}}^b \cdot \bar{\mathbf{v}} \right) d\bar{V} \\ = - \int_{V(t)} \bar{\nabla} \cdot \bar{\mathbf{q}} d\bar{V} + \int_{\partial V(t)} (\bar{\mathbf{v}} \cdot (\bar{\boldsymbol{\sigma}}^{(0)})^T) \cdot d\bar{\mathbf{A}} + \int_{\partial V(t)} ({}^t\bar{\boldsymbol{\Theta}} \cdot (\bar{\mathbf{m}}^{(0)})^T) \cdot d\bar{\mathbf{A}} \\ = - \int_{V(t)} \bar{\nabla} \cdot \bar{\mathbf{q}} d\bar{V} + \int_{V(t)} \bar{\nabla} \cdot (\bar{\mathbf{v}} \cdot (\bar{\boldsymbol{\sigma}}^{(0)})^T) d\bar{V} + \int_{V(t)} \bar{\nabla} \cdot ({}^t\bar{\boldsymbol{\Theta}} \cdot (\bar{\mathbf{m}}^{(0)})^T) d\bar{V} \end{aligned} \quad (195)$$

Following reference [67] one can show that

$$\begin{aligned} \bar{\nabla} \cdot (\bar{\mathbf{v}} \cdot (\bar{\boldsymbol{\sigma}}^{(0)})^T) &= \bar{\mathbf{v}} \cdot (\bar{\nabla} \cdot \bar{\boldsymbol{\sigma}}^{(0)}) + \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \\ \bar{\nabla} \cdot ({}^t\bar{\boldsymbol{\Theta}} \cdot (\bar{\mathbf{m}}^{(0)})^T) &= {}^t\bar{\boldsymbol{\Theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) + \bar{m}_{ji}^{(0)} \frac{{}^t\bar{\Theta}_i}{\partial \bar{x}_j} \end{aligned} \quad (196)$$

Substituting from (196) into (195)

$$\begin{aligned} & \int_{\bar{V}(t)} \left( \bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\rho} \bar{\mathbf{v}} \cdot \frac{D\bar{\mathbf{v}}}{Dt} - \bar{\rho} \bar{\mathbf{F}}^b \cdot \bar{\mathbf{v}} \right) dV \\ &= - \int_{\bar{V}(t)} \bar{\nabla} \cdot \bar{\mathbf{q}} d\bar{V} + \int_{\bar{V}(t)} \left( \bar{\mathbf{v}} \cdot (\bar{\nabla} \cdot \bar{\boldsymbol{\sigma}}^{(0)}) + \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} + {}^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) + \bar{m}_{ji}^{(0)} \frac{\partial {}^t \bar{\theta}_i}{\partial \bar{x}_j} \right) d\bar{V} \quad (197) \end{aligned}$$

Using (169) (balance of linear momenta), (197) reduces to

$$\int_{\bar{V}(t)} \left( \bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\mathbf{q}} - \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{m}_{ji}^{(0)} \frac{\partial {}^t \bar{\theta}_i}{\partial \bar{x}_j} - {}^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \right) dV = 0 \quad (198)$$

Since volume  $\bar{V}$  is arbitrary, the following holds.

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\mathbf{q}} - \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \left( \bar{m}_{ji}^{(0)} \frac{\partial {}^t \bar{\theta}_i}{\partial \bar{x}_j} + {}^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \right) = 0 \quad (199)$$

We note that in the term  ${}^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)})$  we can substitute  $\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}$  from the balance of angular momenta (178), thereby eliminating gradients of  $\bar{\mathbf{m}}^{(0)}$  but instead introducing Cauchy stress tensor  $\bar{\boldsymbol{\sigma}}^{(0)}$ .

#### 4.3.6 Second law of thermodynamics

Let  $\bar{\eta}$  be entropy density in deformed volume  $\bar{V}(t)$ ,  $\bar{h}$  be the entropy flux between  $\bar{V}(t)$  and the volume of matter surrounding it (i.e. contacting sources), and  $\bar{s}$  be the source of entropy in  $\bar{V}(t)$  due to non-contacting bodies, then the rate of increase of entropy in volume  $\bar{V}(t)$  is at least equal to that supplied to  $\bar{V}(t)$  from all contacting and non-contacting sources [67]. Thus

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} d\bar{V} \geq \int_{\partial \bar{V}(t)} \bar{h} d\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} d\bar{V} \quad (200)$$

Cauchy's postulate for  $\bar{h}$  can be stated as

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\mathbf{n}} \quad (201)$$

Using (201) in (200)

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} d\bar{V} \geq - \int_{\partial \bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\mathbf{n}} d\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} d\bar{V} \quad (202)$$

Using  $\bar{\rho} d\bar{V} = \rho_0 dV$  in the left hand side of (202)

$$\frac{D}{Dt} \int_V \eta \rho_0 dV \geq - \int_{\partial V(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\mathbf{n}} d\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} d\bar{V} \quad (203)$$

$$\int_V \frac{D\eta}{Dt} \rho_0 dV \geq - \int_{\partial V(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\mathbf{n}} d\bar{A} + \int_{V(t)} \bar{s} \bar{\rho} d\bar{V} \quad (204)$$

$$\int_{\bar{V}(t)} \bar{\rho} \frac{D\bar{\eta}}{Dt} d\bar{V} \geq - \int_{\bar{V}(t)} \bar{\nabla} \cdot \bar{\boldsymbol{\psi}} d\bar{V} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} d\bar{V} \quad (205)$$

$$\int_{\bar{V}(t)} \left( \bar{\rho} \frac{D\bar{\eta}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{\psi}} - \bar{s} \bar{\rho} \right) d\bar{V} \geq 0 \quad (206)$$

Since the volume  $\bar{V}$  is arbitrary, the following holds.

$$\bar{\rho} \frac{D\bar{\eta}}{Dt} + \bar{\nabla} \cdot \bar{\boldsymbol{\psi}} - \bar{s} \bar{\rho} \geq 0 \quad (207)$$

Inequality (207), referred to as the entropy inequality, is the most fundamental form resulting from the second law of thermodynamics. Inequality (207) is strictly a statement that contains entropy terms, hence contains no information regarding reversible deformation processes such as in case of elastic solids, thus provides no information or mechanisms regarding deriving the constitutive theories for such solids. Only when the mechanical rate of work results in rate of entropy production as in thermofluids and thermoviscoelastic fluids with and without memory will inequality (207) have some information regarding the associated conjugate pairs that result in rate of entropy production. We also note that (207) in its present form does not provide any information regarding constitutive theory for heat vector  $\bar{\mathbf{q}}$ .

An alternate form of the entropy inequality is possible using a relationship between  $\bar{\boldsymbol{\psi}}$  and  $\bar{\mathbf{q}}$  and the energy equation. Since the energy equation has all possible mechanisms that result in energy storage and dissipation, this form of entropy inequality derived using the energy equation is expected to be helpful in the derivations of constitutive theories. Using

$$\bar{\boldsymbol{\psi}} = \frac{\bar{\mathbf{q}}}{\bar{\theta}}, \quad \bar{s} = \frac{\bar{r}}{\bar{\theta}} \quad (208)$$

where  $\bar{\theta}$  is absolute temperature,  $\bar{\mathbf{q}}$  is heat vector, and  $\bar{r}$  is a suitable potential, then

$$\bar{\nabla} \cdot \bar{\boldsymbol{\psi}} = \bar{\psi}_{i,i} = \frac{\bar{q}_{i,i}}{\bar{\theta}} - \frac{\bar{q}_i \bar{\theta}_{,i}}{\bar{\theta}^2} = \frac{\bar{q}_{i,i}}{\bar{\theta}} - \frac{\bar{q}_i \bar{g}_i}{\bar{\theta}^2} = \frac{\bar{\nabla} \cdot \bar{\mathbf{q}}}{\bar{\theta}} - \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}^2} \quad (209)$$

Substituting from (208) and (209) into (207) and multiplying throughout by  $\bar{\theta}$  yields

$$\bar{\rho} \bar{\theta} \frac{D\bar{\eta}}{Dt} + (\bar{\nabla} \cdot \bar{\mathbf{q}} - \bar{\rho} \bar{r}) - \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} \geq 0 \quad (210)$$

From the energy Equation (199) (after inserting  $\bar{\rho} \bar{r}$  term)

$$\bar{\nabla} \cdot \bar{\mathbf{q}} - \bar{\rho} \bar{r} = -\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} + \bar{m}_{ji}^{(0)} \frac{{}^t \bar{\theta}_i}{\partial \bar{x}_j} + {}^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \quad (211)$$

Substituting (211) in (210)

$$\bar{\rho}\bar{\theta}\frac{D\bar{\eta}}{Dt} - \bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\sigma}_{ji}^{(0)}\frac{\partial\bar{v}_i}{\partial\bar{x}_j} + \bar{m}_{ji}^{(0)}\frac{{}^t\bar{\theta}_i}{\partial\bar{x}_j} + {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) - \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} \geq 0 \quad (212)$$

or

$$\bar{\rho}\left(\frac{D\bar{e}}{Dt} - \bar{\theta}\frac{D\bar{\eta}}{Dt}\right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \bar{\sigma}_{ji}^{(0)}\frac{\partial\bar{v}_i}{\partial\bar{x}_j} - \bar{m}_{ji}^{(0)}\frac{{}^t\bar{\theta}_i}{\partial\bar{x}_j} - {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (213)$$

Let  $\bar{\Phi}$  be the Helmholtz free energy density defined by

$$\bar{\Phi} = \bar{e} - \bar{\eta}\bar{\theta} \quad (214)$$

Then we have

$$\frac{D\bar{e}}{Dt} - \bar{\theta}\frac{D\bar{\eta}}{Dt} = \frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt} \quad (215)$$

Substituting (215) into (213) gives

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \bar{\sigma}_{ji}^{(0)}\frac{\partial\bar{v}_i}{\partial\bar{x}_j} - \bar{m}_{ji}^{(0)}\frac{{}^t\bar{\theta}_i}{\partial\bar{x}_j} - {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (216)$$

or

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr}\left([\bar{\boldsymbol{\sigma}}^{(0)}]\left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right]\right) - \text{tr}\left([\bar{\mathbf{m}}^{(0)}]\left[\frac{\partial\{{}^t\bar{\theta}\}}{\partial\{\bar{x}\}}\right]\right) - {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (217)$$

or

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr}\left([\bar{\boldsymbol{\sigma}}^{(0)}][\bar{L}]\right) - \text{tr}\left([\bar{\mathbf{m}}^{(0)}][{}^t\bar{\boldsymbol{\theta}}J]\right) - {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (218)$$

in which  $[{}^t\bar{\boldsymbol{\theta}}J]$  is the gradient of total rotation rates. Inequality (218) resulting from the second law of thermodynamics is the most fundamental form of entropy inequality in Helmholtz free energy density  $\bar{\Phi}$ . A slightly more expanded form of (218) that is more useful in the derivations of the constitutive theories can be derived using the decomposition of various tensors in (218) into symmetric and antisymmetric tensors, presented in a later section.

Noting that

$${}^t\bar{\boldsymbol{\theta}} = {}^t_i\bar{\boldsymbol{\theta}} - {}^t_e\bar{\boldsymbol{\theta}} \quad (219)$$

$$[{}^t\bar{\boldsymbol{\theta}}J] = [{}^t_i\bar{\boldsymbol{\theta}}J] - [{}^t_e\bar{\boldsymbol{\theta}}J] = \left[\frac{\partial\{{}^t_i\bar{\theta}\}}{\partial\{\bar{x}\}}\right] - \left[\frac{\partial\{{}^t_e\bar{\theta}\}}{\partial\{\bar{x}\}}\right] \quad (220)$$

and from balance of angular momenta (178)

$$\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)} = \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} \quad (221)$$

and from (28)

$$[\bar{L}] = [\bar{\mathbb{L}}] + [{}_e\bar{W}] \quad (222)$$

Using (219) – (222) in (218), we obtain

$$\bar{\rho} \left( \frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr} \left( [\bar{\sigma}^{(0)}]([\bar{\mathbb{L}}] + [{}_e\bar{W}]) \right) - \text{tr} \left( [\bar{m}^{(0)}][{}^t\bar{\theta}\bar{\mathbb{J}}] \right) - ({}^t\bar{\theta} - {}^t_e\bar{\theta}) \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (223)$$

or

$$\begin{aligned} & \bar{\rho} \left( \frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr} \left( [\bar{\sigma}^{(0)}][\bar{\mathbb{L}}] \right) - \text{tr} \left( [\bar{\sigma}^{(0)}][{}_e\bar{W}] \right) \\ & - \text{tr} \left( [\bar{m}^{(0)}][{}^t\bar{\theta}\bar{\mathbb{J}}] \right) - {}^t_i\bar{\theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) + {}^t_e\bar{\theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \end{aligned} \quad (224)$$

A simple calculation shows that

$$\text{tr} \left( [\bar{\sigma}^{(0)}][{}_e\bar{W}] \right) = {}^t_e\bar{\theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \quad (225)$$

Using (225) in (224), (224) reduces to

$$\bar{\rho} \left( \frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr} \left( [\bar{\sigma}^{(0)}][\bar{\mathbb{L}}] \right) - \text{tr} \left( [\bar{m}^{(0)}][{}^t\bar{\theta}\bar{\mathbb{J}}] \right) - {}^t_i\bar{\theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (226)$$

Inequality (226) is the desired form of entropy inequality that is useful in deriving constitutive theories. By making similar substitutions and simplifications, the energy Equation (199) can be written as

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\mathbf{q}} - \text{tr} \left( [\bar{\sigma}^{(0)}][\bar{\mathbb{L}}] \right) - \text{tr} \left( [\bar{m}^{(0)}][{}^t\bar{\theta}\bar{\mathbb{J}}] \right) - {}^t_i\bar{\theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) = 0 \quad (227)$$

#### 4.3.7 Rate of work conjugate pairs in the entropy inequality

As well known, determination of conjugate pairs either using energy equation or entropy inequality is essential in deriving constitutive theories. From the entropy inequality it may appear that  $([\bar{\sigma}^{(0)}], [\bar{\mathbb{L}}])$  and  $([\bar{m}^{(0)}], [{}^t\bar{\theta}\bar{\mathbb{J}}])$  are rate of work conjugate pairs. The additional term  ${}^t_i\bar{\theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)})$  also needs to be accounted for. Also  $(\bar{\mathbf{q}}, \bar{\mathbf{g}})$  appear to be a conjugate pair. Once the rate of work conjugate pairs are established from the energy equation or entropy inequality, the constitutive theories can be derived using the theory of generators and invariants, i.e. representation theorem in conjunction with entropy inequality.

We note that  $[\bar{\sigma}^{(0)}]$  and  $[\bar{\mathbb{L}}]$  are both non-symmetric tensors whereas  $[\bar{m}^{(0)}]$  is a symmetric tensor (when balance of moments of moments is used as a balance law as we have done in this paper) but  $[{}^t\bar{\theta}\bar{\mathbb{J}}]$  is a non-symmetric tensor. Whether  $([\bar{\sigma}^{(0)}], [\bar{\mathbb{L}}])$  and  $([\bar{m}^{(0)}], [{}^t\bar{\theta}\bar{\mathbb{J}}])$  are actually rate of work conjugate needs to be established.

Consider entropy inequality (226). Decompose  $\bar{\sigma}^{(0)}$  into symmetric ( ${}_s\bar{\sigma}^{(0)}$ ) and anti-symmetric ( ${}_a\bar{\sigma}^{(0)}$ ) tensors and we also use decomposition of  $\bar{\mathbb{L}}$  into symmetric and skew symmetric tensors as in (26). Also decompose  ${}^t\bar{\theta}\bar{\mathbb{J}}$  into symmetric ( ${}^t_s\bar{\theta}\bar{\mathbb{J}}$ ) and antisymmetric ( ${}^t_a\bar{\theta}\bar{\mathbb{J}}$ ) tensors and substitute these into the entropy inequality (226).

$$\bar{\sigma}^{(0)} = {}_s\bar{\sigma}^{(0)} + {}_a\bar{\sigma}^{(0)} \quad (228)$$



$${}^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}} = {}^t\bar{\boldsymbol{\sigma}}_s\bar{\mathbf{J}} + {}^t\bar{\boldsymbol{\sigma}}_a\bar{\mathbf{J}} \quad (229)$$

$$\begin{aligned} & \bar{\rho} \left( \frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr} \left( ([{}_s\bar{\boldsymbol{\sigma}}^{(0)}] + [{}_a\bar{\boldsymbol{\sigma}}^{(0)}])([\bar{\mathbf{D}}] + [{}_t\bar{\mathbf{W}}]) \right) \\ & - \text{tr} \left( [\bar{\mathbf{m}}^{(0)}]([{}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}] + [{}_a^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}]) \right) - {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \end{aligned} \quad (230)$$

Since

$$\text{tr} \left( [{}_s\bar{\boldsymbol{\sigma}}^{(0)}][{}_a\bar{\mathbf{W}}] \right) = 0 \quad (231)$$

$$\text{tr} \left( [{}_a\bar{\boldsymbol{\sigma}}^{(0)}][\bar{\mathbf{D}}] \right) = 0 \quad (232)$$

$$\text{tr} \left( [\bar{\mathbf{m}}^{(0)}][{}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}] \right) = 0 \quad (233)$$

entropy inequality (230) can be written as

$$\begin{aligned} & \bar{\rho} \left( \frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr} \left( [{}_s\bar{\boldsymbol{\sigma}}^{(0)}][\bar{\mathbf{D}}] \right) - \text{tr} \left( [{}_a\bar{\boldsymbol{\sigma}}^{(0)}][{}_t\bar{\mathbf{W}}] \right) - \text{tr} \left( [\bar{\mathbf{m}}^{(0)}][{}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}] \right) - {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \\ & \leq 0 \end{aligned} \quad (234)$$

In (234)  $\bar{\mathbf{q}}$  and  $\bar{\mathbf{g}}$ ,  $[{}_s\bar{\boldsymbol{\sigma}}^{(0)}]$  and  $[\bar{\mathbf{D}}]$ ,  $[{}_a\bar{\boldsymbol{\sigma}}^{(0)}]$  and  $[{}_t\bar{\mathbf{W}}]$ ,  $[\bar{\mathbf{m}}^{(0)}]$  and  $[{}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}]$  are the rate of work conjugate pairs that are in conformity with the works of Spencer, Wang, Zheng, etc [73–89], i.e. symmetric tensors are conjugate with symmetric tensors and antisymmetric tensors are conjugate with antisymmetric tensors. Using (228), (229), and (231) – (233), the energy equation can also be written as

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\mathbf{q}} - \text{tr} \left( [{}_s\bar{\boldsymbol{\sigma}}^{(0)}][\bar{\mathbf{D}}] \right) - \text{tr} \left( [{}_a\bar{\boldsymbol{\sigma}}^{(0)}][{}_t\bar{\mathbf{W}}] \right) - \text{tr} \left( [\bar{\mathbf{m}}^{(0)}][{}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}] \right) - {}^t\bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) = 0 \quad (235)$$

From the conjugate pairs it is straightforward to conclude that for the simplest possible case

$$\begin{aligned} {}_s\bar{\boldsymbol{\sigma}}^{(0)} &= {}_s\bar{\boldsymbol{\sigma}}^{(0)}(\bar{\mathbf{D}}, \bar{\theta}) \\ {}_a\bar{\boldsymbol{\sigma}}^{(0)} &= {}_a\bar{\boldsymbol{\sigma}}^{(0)}({}_t\bar{\mathbf{W}}, \bar{\theta}) \\ \bar{\mathbf{m}}^{(0)} &= \bar{\mathbf{m}}^{(0)}({}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}, \bar{\theta}) \\ \bar{\mathbf{q}} &= \bar{\mathbf{q}}(\bar{\mathbf{g}}, \bar{\theta}) \end{aligned} \quad (236)$$

Furthermore, one also concludes that  ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$  is not a function of  ${}_t\bar{\mathbf{W}}$  and  ${}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}$ ,  ${}_a\bar{\boldsymbol{\sigma}}^{(0)}$  is not a function of  $\bar{\mathbf{D}}$  and  ${}_s^t\bar{\boldsymbol{\sigma}}\bar{\mathbf{J}}$ , and  $\bar{\mathbf{m}}^{(0)}$  is not a function of  $\bar{\mathbf{D}}$  and  ${}_t\bar{\mathbf{W}}$ .

#### 4.3.8 Final form of conservation and balance laws

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\nabla} \cdot (\bar{\rho} \bar{\mathbf{v}}) = 0 \quad (237)$$

$$\bar{\rho} \frac{D\bar{\mathbf{v}}}{Dt} - \bar{\rho} \bar{\mathbf{F}}^b - \bar{\nabla} \cdot \bar{\boldsymbol{\sigma}}^{(0)} = 0 \quad (238)$$

$$\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)} - \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = 0 \quad (239)$$

$$\epsilon_{ijk} \bar{\mathbf{m}}_{ij}^{(0)} = 0 \quad (240)$$

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\mathbf{q}} - \text{tr} \left( [{}_s \bar{\boldsymbol{\sigma}}^{(0)}] [\bar{D}] \right) - \text{tr} \left( [{}_a \bar{\boldsymbol{\sigma}}^{(0)}] [{}_t \bar{W}] \right) - \text{tr} \left( [\bar{\mathbf{m}}^{(0)}] [{}_s^t \bar{J}] \right) - {}_i^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) = 0 \quad (241)$$

$$\bar{\rho} \left( \frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr} \left( [{}_s \bar{\boldsymbol{\sigma}}^{(0)}] [\bar{D}] \right) - \text{tr} \left( [{}_a \bar{\boldsymbol{\sigma}}^{(0)}] [{}_t \bar{W}] \right) - \text{tr} \left( [\bar{\mathbf{m}}^{(0)}] [{}_s^t \bar{J}] \right) - {}_i^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (242)$$

In mathematical model (237)–(242), the dependent variables are:  $\bar{\rho}$  (1),  $\bar{\mathbf{v}}$  (3),  ${}_s \bar{\boldsymbol{\sigma}}^{(0)}$  (6),  ${}_a \bar{\boldsymbol{\sigma}}^{(0)}$  (3),  $\bar{\mathbf{m}}^{(0)}$  (6),  $\bar{\mathbf{q}}$  (3),  $\bar{\theta}$  (1), and  ${}_i^t \bar{\boldsymbol{\theta}}$  (3), a total of 26.  $\bar{\Phi}$ ,  $\bar{e}$ , and  $\bar{\eta}$  are not dependent variables as these can be expressed in terms of others. Numbers in brackets refer to the number of variables. The equations in the model are: continuity (1), linear momentum (3), angular momentum (3), energy (1), constitutive theories for  ${}_s \bar{\boldsymbol{\sigma}}^{(0)}$  (6),  ${}_a \bar{\boldsymbol{\sigma}}^{(0)}$  (3),  $\bar{\mathbf{m}}^{(0)}$  (6), and  $\bar{\mathbf{q}}$  (3), a total of 26, hence the mathematical model has closure.

#### 4.3.9 Constitutive theories [97,98]

The general discussion and steps (1) – (3) presented in [section 4.1.9](#) hold here as well, hence not repeated. In the following we give final forms of the entropy inequality, the constitutive variables and their argument tensors for thermoviscous fluids and thermo-viscoelastic fluids.

(a) Thermoviscous fluids [97]

we have

$${}_s \bar{\boldsymbol{\sigma}}^{(0)} = {}_e({}_s \bar{\boldsymbol{\sigma}}^{(0)}) + {}_d({}_s \bar{\boldsymbol{\sigma}}^{(0)})$$

in which

$${}_e({}_s \bar{\boldsymbol{\sigma}}^{(0)}) = \bar{p}(\bar{\theta}) \mathbf{I}; \quad \text{incompressible}$$

$${}_e({}_s \bar{\boldsymbol{\sigma}}^{(0)}) = \bar{p}(\bar{\rho}, \bar{\theta}) \mathbf{I}; \quad \text{compressible}$$

The entropy inequality reduces to

$$\frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - {}_d({}_s \bar{\boldsymbol{\sigma}}^{(0)})_{ik} \bar{D}_{ik} - \text{tr} \left( [{}_a \bar{\boldsymbol{\sigma}}^{(0)}] [{}_t \bar{W}] \right) - \text{tr} \left( [\bar{\mathbf{m}}^{(0)}] [{}_s^t \bar{J}] \right) - {}_i^t \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (243)$$

The entropy inequality (243) is satisfied if

$$\begin{aligned} {}_s \psi_d &= ({}_d({}_s \bar{\boldsymbol{\sigma}}^{(0)})_{ik}) (\bar{D}_{ik}) > 0 \\ {}_a \psi_d &= ({}_a \bar{\boldsymbol{\sigma}}^{(0)})_{ik} ({}_t \bar{W}_{ik}) > 0 \\ {}_m \psi_d &= (\bar{\mathbf{m}}_{ik}^{(0)}) ({}_s^t \bar{J}_{ik}) \geq 0 \end{aligned} \quad (244)$$

$$\frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - {}^t_i \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) \leq 0 \quad (245)$$

To ensure inequality (245) is satisfied the following must hold.

$$\frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} \leq 0 \quad (246)$$

$${}^t_i \bar{\boldsymbol{\theta}} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)}) = 0 \quad (247)$$

The inequalities in (244) imply that the rate of work due to  ${}_d({}_s \bar{\boldsymbol{\sigma}}^{(0)})$ ,  ${}_a \bar{\boldsymbol{\sigma}}^{(0)}$  and  $\bar{\mathbf{m}}^{(0)}$  (i.e.  ${}_s \psi_d$ ,  ${}_a \psi_d$  and  ${}_m \psi_d$ ) must be positive.

The constitutive variables and their arguments are

$$\begin{aligned} \Phi &= \Phi(\bar{\rho}, \bar{\theta}) \\ {}_d({}_s \bar{\boldsymbol{\sigma}}^{(0)}) &= {}_d({}_s \bar{\boldsymbol{\sigma}}^{(0)})(\bar{\rho}, \mathbf{y}_{(k)}; k = 1, 2, \dots, n, \bar{\theta}) \\ \bar{\mathbf{m}}^{(0)} &= \bar{\mathbf{m}}^{(0)}(\bar{\rho}, {}^t_s \bar{\mathbf{J}}, \bar{\theta}) \\ \bar{\mathbf{q}} &= \bar{\mathbf{q}}(\bar{\mathbf{g}}, \bar{\theta}) \end{aligned} \quad (248)$$

Constitutive theories and material coefficients are derived using representation theorem and Taylor series expansion. This constitutive theory is ordered rate theory of upto order  $n$ , the order of the strain tensor.

(b) Thermoviscoelastic fluids [98]

$${}_s \bar{\boldsymbol{\sigma}}^{(0)} = {}_e({}_s \bar{\boldsymbol{\sigma}}^{(0)}) + {}_d({}_s \bar{\boldsymbol{\sigma}}^{(0)})$$

In this case also the entropy inequality (243) and condition (244) – (247) hold. The constitutive variables and their argument tensors are

$$\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \quad (249)$$

$${}_d({}_s \bar{\boldsymbol{\sigma}}^{(m_{s\sigma})}) = {}_d({}_s \bar{\boldsymbol{\sigma}}^{(m_{s\sigma})})(\bar{\rho}, \mathbf{y}_{(i)}; i = 1, 2, \dots, n, {}_d({}_s \bar{\boldsymbol{\sigma}}^{(j)}); j = 0, 1, \dots, (m_{s\sigma} - 1), \bar{\theta}) \quad (250)$$

$$\bar{\mathbf{m}}^{(m_m)} = \bar{\mathbf{m}}^{(m_m)}(\bar{\rho}, {}^t_s \bar{\mathbf{J}}, \bar{\mathbf{m}}^{(j)}; j = 0, 1, \dots, (m_m - 1), \bar{\theta}) \quad (251)$$

$${}_a \bar{\boldsymbol{\sigma}}^{(m_{a\sigma})} = {}_a \bar{\boldsymbol{\sigma}}^{(m_{a\sigma})}(\bar{\rho}, {}^t \bar{\mathbf{W}}, {}_a \bar{\boldsymbol{\sigma}}^{(k)}; k = 0, 1, \dots, (m_{a\sigma} - 1), \bar{\theta}) \quad (252)$$

$$\bar{\mathbf{q}} = \bar{\mathbf{q}}(\bar{\mathbf{g}}, \bar{\theta}) \quad (253)$$

Constitutive theories and the material coefficients are derived using representation theorem and Taylor series expansion. Relaxation modulus can be derived and the models such as Maxwell, Oldroyd-B and Giesekus are a subset of the constitutive theories presented here (see Surana et al. [98] for complete details). The resulting constitutive theories are ordered rate theories of upto orders  $n$ ,  $m_m$  and  $m_{a\sigma}$  corresponding to  $\mathbf{y}_{(i)}$ ,  $\bar{\mathbf{m}}^{(j)}$  and  ${}_a \bar{\boldsymbol{\sigma}}^{(k)}$ ;  $i = 1, 2, \dots, n$ ;  $j = 0, 1, \dots, m_m$  and  $k = 0, 1, \dots, m_{a\sigma}$ .

#### 4.4 Conservation and balance laws for fluent continua incorporating internal rotation rates only

The conservation and the balance laws for this case can be deduced from those presented for fluent continua with internal and Cosserat rotation rates by letting in (237) – (242)

$$[_t\bar{W}] = [\bar{W}], \quad [{}^t\bar{\Theta}J] = [{}^t\bar{\Theta}J]$$

$$\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)} = \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}$$

and

$$\text{tr}[_a\bar{\boldsymbol{\sigma}}^{(0)}][\bar{W}] = -{}_t\bar{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)})$$

and we obtain

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\nabla} \cdot (\bar{\rho} \bar{\mathbf{v}}) = 0 \quad (254)$$

$$\bar{\rho} \frac{D\bar{\mathbf{v}}}{Dt} - \bar{\rho} \bar{\mathbf{F}}^b - \bar{\nabla} \cdot \bar{\boldsymbol{\sigma}}^{(0)} = 0 \quad (255)$$

$$\bar{\nabla} \cdot \bar{\mathbf{m}}^{(0)} - \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = 0 \quad (256)$$

$$\epsilon_{ijk} \bar{m}_{ij}^{(0)} = 0 \quad (257)$$

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot \bar{\mathbf{q}} - \text{tr}([{}_s\bar{\boldsymbol{\sigma}}^{(0)}][\bar{D}]) - \text{tr}([\bar{\mathbf{m}}^{(0)}][{}^t\bar{\Theta}J]) = 0 \quad (258)$$

$$\bar{\rho} \left( \frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\Theta}}{Dt} \right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \text{tr}([{}_s\bar{\boldsymbol{\sigma}}^{(0)}][\bar{D}]) - \text{tr}([\bar{\mathbf{m}}^{(0)}][{}^t\bar{\Theta}J]) \leq 0 \quad (259)$$

In mathematical model (254) – (259), the dependent variables are:  $\bar{\rho}$  (1),  $\bar{\mathbf{v}}$  (3),  ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$  (6),  ${}_a\bar{\boldsymbol{\sigma}}^{(0)}$  (3),  $\bar{\mathbf{m}}^{(0)}$  (6),  $\bar{\mathbf{q}}$  (3) and  $\bar{\Theta}$  (1), a total of 23.  $\bar{\Phi}$ ,  $\bar{e}$ , and  $\bar{\eta}$  are not dependent variables as these can be expressed in terms of others. Numbers in brackets refer to the number of variables. The equations in the model are: linear momentum (3), angular momentum (3), energy (1), constitutive theories for  ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$  (6), conservation of mass (1),  $\bar{\mathbf{m}}^{(0)}$  (6), and  $\bar{\mathbf{q}}$  (3), a total of 23, hence the mathematical model has closure.

This mathematical model has also been derived by Surana et al. [99,100] from basic principles using internal rotation rates only. In the following we present some details of the constitutive variables and their argument tensors. These can be derived following the procedure outlined in section 4.1.9 for solid continua or by following Surana et al. [67].

a) Thermoviscous fluids [101]

$${}_s\bar{\boldsymbol{\sigma}}^{(0)} = {}_e({}_s\bar{\boldsymbol{\sigma}}^{(0)}) + {}_d({}_s\bar{\boldsymbol{\sigma}}^{(0)}) \quad (260)$$

in which

$$\begin{aligned}
 & e({}_s\bar{\boldsymbol{\sigma}}^{(0)}) = \bar{p}(\bar{\boldsymbol{\theta}})\mathbf{I}; \quad \text{incompressible} \\
 \text{and} \quad & d({}_s\bar{\boldsymbol{\sigma}}^{(0)}) = \bar{p}(\bar{\rho}, \bar{\boldsymbol{\theta}})\mathbf{I}; \quad \text{compressible}
 \end{aligned} \tag{261}$$

entropy inequality reduces to

$$\frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - \text{tr}([d({}_s\bar{\boldsymbol{\sigma}}^{(0)})][\bar{D}]) - \text{tr}([\bar{m}^{(0)}][{}^{\bar{\theta}}_s \bar{J}]) \leq 0 \tag{262}$$

Entropy inequality (262) is satisfied if

$$\begin{aligned}
 & \frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} \leq 0, \quad \text{tr}([d({}_s\bar{\boldsymbol{\sigma}}^{(0)})][\bar{D}]) > 0 \\
 & \text{tr}([\bar{m}^{(0)}][{}^{\bar{\theta}}_s \bar{J}]) > 0
 \end{aligned} \tag{263}$$

The constitutive variables and their arguments are

$$\begin{aligned}
 \Phi &= \Phi(\bar{\rho}, \bar{\boldsymbol{\theta}}) \\
 d({}_s\bar{\boldsymbol{\sigma}}^{(0)}) &= d({}_s\bar{\boldsymbol{\sigma}}^{(0)})(\bar{\rho}, \boldsymbol{\gamma}_{(k)}; k = 1, 2, \dots, n, \bar{\boldsymbol{\theta}}) \\
 \bar{m}^{(0)} &= \bar{m}^{(0)}(\bar{\rho}, {}^{\bar{\theta}}_s \bar{J}, \bar{\boldsymbol{\theta}}) \\
 \bar{q} &= \bar{q}(\bar{g}, \bar{\boldsymbol{\theta}})
 \end{aligned} \tag{264}$$

The resulting constitutive theory is ordered rate theory of upto orders  $n$  in of the strain rate tensors.

b) Thermoviscoelastic fluids [102]

In this case also (260) – (263) remain the same but the constitutive variables and their argument tensors change.

$$\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\boldsymbol{\theta}})$$

$$d({}_s\bar{\boldsymbol{\sigma}}^{(m_s\sigma)}) = d({}_s\bar{\boldsymbol{\sigma}}^{(m_s\sigma)})(\bar{\rho}, \boldsymbol{\gamma}_{(i)}; i = 1, 2, \dots, n, d({}_s\bar{\boldsymbol{\sigma}}^{(j)}); j = 0, 1, \dots, (m_s - 1), \bar{\boldsymbol{\theta}})$$

$$\bar{m}^{(m_m)} = \bar{m}^{(m_m)}(\bar{\rho}, {}^{\bar{\theta}}_s \bar{J}, \bar{m}^{(k)}; k = 0, 1, \dots, (m_m - 1), \bar{\boldsymbol{\theta}})$$

$$\bar{q} = \bar{q}(\bar{g}, \bar{\boldsymbol{\theta}})$$

Constitutive theories and the material coefficients for both (a) and (b) are derived using representation theorem and Taylor series expansion. The resulting constitutive theories are ordered rate theories of upto orders  $n$ ,  $m_{s\sigma}$  and  $m_m$  of  $\boldsymbol{\gamma}_{(i)}$ ,  $d({}_s\bar{\boldsymbol{\sigma}}^{(j)})$  and  $\bar{m}^{(k)}$ ;  $i = 1, 2, \dots, n$ ;  $j = 0, 1, \dots, m_{s\sigma}$  and  $k = 0, 1, \dots, m_m$ .

## 5 General remarks

In this section we make some remarks regarding the non-classical continuum theories for solid continua incorporating internal rotations due to antisymmetric part of the gradient of deformation and a non-classical continuum theory incorporating internal as well as Cosserat rotations. For fluent continua we make some remarks regarding non-classical continuum theories incorporating internal rotation rates due to antisymmetric part of the velocity gradient tensor and theories incorporating internal rotation rates as

well as Cosserat rotation rates. For all four non-classical continuum theories constitutive theories are also presented: (i) for thermoelastic, thermoviscoelastic without memory and thermoviscoelastic solid continua and (ii) thermoviscous and thermoviscoelastic for fluent continua.

- (1) All non-classical continuum theories presented here are only valid for homogeneous and isotropic matter. These assumptions are rather fundamental in continuum mechanics for the differential form of the conservation and the balance laws.
- (2) The non-classical continuum theory for solid continua incorporating internal rotations is motivated by the fact that, since Jacobian of deformation (or displacement gradient tensor) is fundamental and complete measure of deformation physics in the classical continuum mechanics, it must be incorporated in its entirety in the derivation of the conservation and the balance laws. Decomposition of  ${}^d\mathbf{J} = {}_s^d\mathbf{J} + {}_a^d\mathbf{J}$  clearly suggests that this can be accomplished by incorporating  ${}_a^d\mathbf{J}$  (rotations) in the currently used classical continuum mechanics conservation and balance laws. Thus, the non-classical continuum theory with internal rotations is based on additional deformation physics in  ${}_a^d\mathbf{J}$  that is neglected in classical continuum mechanics. Thus, this theory corrects a major deficiency in the classical continuum theory. The Cauchy moment tensor results due to resistance provided by deforming matter to the varying rotations between the material points.
- (3) Similar argument as in (2) for solid continua holds for non-classical continuum theories presented here for fluent continua based on internal rotation rates that results in Cauchy moment tensor when the rotation rates are resisted by the deforming fluent continua.
- (4) The internal rotations and the internal rotation rates are completely defined by  ${}_a^d\mathbf{J}$  and  $\mathbf{W}$ , hence are not additional degrees of freedom at a material point in the solid and the fluent continua.
- (5) Cosserat rotations and the Cosserat rotation rates are additional three degrees of freedom at a material point in solid and fluent continua. Thus, in a non-classical solid continuum with internal and Cosserat rotations a material point has six degrees of freedom,  $\mathbf{u}$  and  ${}_e\mathbf{\Theta}$ . Likewise in fluent continua we have  $\bar{\mathbf{v}}$  and  ${}_e^t\mathbf{\Theta}$  as six degrees of freedom at a material point. The additional three degrees of freedom in solid and fluent continua due to Cosserat rotations and Cosserat rotation rates are incorporated so that a more comprehensive description of the physics may be facilitated using the resulting thermodynamic framework.
- (6) The non-classical continuum theories and the constitutive theories presented here are consistent and more comprehensive thermodynamic frameworks for homogeneous and isotropic solid and fluent continua.
- (7) The derivation presented in this paper does not consider microconstituent, micro-deformation as well as associated kinematic considerations. What physics can possibly be described using the theories presented here is obviously of interest. In the applications of such theories one must always keep in mind that these theories in their present form can not be applied to nonhomogeneous and nonisotropic continua without some modifications that may be warranted by such physics.

- (8) We do remark that whether microconstituent and microdeformation considerations can be approximated in some manner to make use of such theories with appropriate modifications of the balance laws presented here is worthy of investigation. We elaborate more on (6) – (8) in the next section.
- (9) Consideration of new physics in these theories due to internal and Cosserat rotations and their rates requires additional balance law due to the fact that this physics is not considered in the classical continuum theories. We have presented rationale for the new balance law, balance of moment of moments and have given its derivation based on rate considerations as required for a balance law.
- (10) In all four non-classical continuum theories rotation or rotation rate tensors when resisted by deforming continua result in Cauchy moment tensor. The internal rotations or the internal rotation rates and the conjugate moment tensor physics is always present in all deforming solid and fluent continua. The physics due to Cosserat rotations and the Cosserat rotation rates is additional physics. Even though both internal and Cosserat rotations as well as their rates act about the axes of the same triad at a material point, the constitutive considerations for the corresponding Cauchy moment tensors may contain different material coefficients.
- (11) We note that when both the internal as well as Cosserat rotations exist, the compatibility equation needs to be satisfied as we have shown. This is a consistency condition on the deformation physics.
- (12) In the non-classical theories presented here, the Cauchy stress tensor is always non-symmetric and is balanced by the gradients of the Cauchy moment tensor through the balance of angular momenta, but the Cauchy moment tensor is symmetric when the balance of moment of moments is considered as a balance law. Surana et al. [71,72] have shown that in the absence of this balance law, the nonsymmetry of Cauchy moment tensor requires additional constitutive theory for the antisymmetric Cauchy moment tensor which results in spurious and non-physical deformation physics.

## **6. Non-classical continuum theories presented in this paper, micropolar, microstretch and micromorphic theories of Eringen**

We provide some discussion on the exhaustive work by Eringen [2–14,20,21] on micropolar, microstretch and micromorphic polar continuum theories specially in context with the non-classical continuum theories presented in this paper. These works consider a volume of matter with microconstituents. The microconstituents locations in the volume are uniquely identified by their coordinates with respect to the coordinates of the center of mass of the volume. As the volume deforms the center of mass displaces and the constituents displace relative to the center of mass. The deformation physics of microconstituents is established using classical continuum mechanics i.e. measures of length, change in length, strain measures, etc. Microdisplacement tensor is introduced and the further developments eventually leads to introduction of internal rotations and unknown rotations similar to Cosserat rotations in the final theory.

An important question is *Is this theory same as the non-classical continuum theories presented in this paper that considers internal rotations and unknown Cosserat rotations?* We make following remarks to answer this question.

- (1) The non-classical continuum theories presented in this paper are only applicable to homogeneous and isotropic solid and fluent continua.
- (2) If each microconstituent is uniquely identified in the volume of matter, then obviously there are countable number of them in the volume considered. Consideration of the deformation physics of the microconstituents clearly establishes that their deformation influences the behavior of the volume of matter. Under these conditions the volume of matter can not be treated as homogeneous and isotropic. Consideration of such physics requires that the conservation and the balance laws perhaps be considered in the integral forms as the integrand is not invariant of the volume of matter considered. In other words in this case we only have integral form of the conservation and balance laws.
- (3) Due to the presence of microconstituents in the volume of matter we no longer have isotropic and homogeneous volume of matter. Use of classical continuum mechanics in establishing the deformation physics of the microconstituents requires additional considerations. Even though use of classical continuum mechanics for an isolated microfiber maybe valid, however interaction of this physics with the matrix is essential to establish how the entire volume is affected by each microconstituent. The validity of the concept of length, its deformation based on classical continuum mechanics for isotropic homogeneous matter when applied to the volume of matter with microconstituents is of serious concern as well.
- (4) In our work we do not have any of these concerns. The non-classical continuum theories presented here are based on internal and Cosserat rotations and their rates are only valid for isotropic homogeneous matter in their present form.
- (5) Use of classical continuum mechanics for microconstituent deformation physics also requires some additional consideration. If the mean free path between the molecules of the matrix material is much smaller than the size of the microconstituents ( $O(10^{-6})$  meters), then the use of classical continuum mechanics for microconstituent deformation physics is valid. Thus, in case of metals, ceramics, incompressible fluids in which the mean free path is of the order of  $O(10^{-10})$  meters or smaller, this is valid. However, in cases of gases with mean free path generally of the order of  $O(10^{-6})$  meters, use of classical continuum mechanics for establishing deformation physics of microconstituents is not valid. It is for the same reason that nano particles ( $O(10^{-9})$  meters) deformation physics can not be described using classical continuum mechanics.

## 7. Non-classical continuum theories presented in this paper and couple stress theories

The couple stress theories assume existence of average moment on the oblique plane of the deformed tetrahedron which through Cauchy principle gives rise to Cauchy moment tensor. The Cauchy moment tensor can be assumed to be conjugate to internal rotations



or Cosserat rotations or both giving rise to different theories. The major draw back of this approach is that the assumption of average moment tensor is not a kinematic consideration, hence suffers from non-uniqueness in the decision regarding the conjugate kinematic quantity.

In the present work, internal rotation or Cosserat rotation or both are kinematic considerations for solid continua. Same is true for internal rotation rates or Cosserat rotation rates or both for fluent continua. They all result in rate of work conjugate moment. In the non-classical continuum theories presented here kinematic consideration (choice of the type of rotation or rotation rate) establish the physics contained in the development of the non-classical continuum theory, rest follows from the consistent thermodynamic principles of continuum mechanics and constitute theories based on representation theorem.

The couple stress theories have led to the belief that there is additional length scale involved in the description of the physics of the deforming matter. The assumption of isotropic, homogeneous matter is fundamental in the non-classical theories we have presented. We have shown that internal rotation or rotation rate in the non-classical continuum theories incorporate physics due to  ${}^d\mathbf{J}$  or  $\bar{\mathbf{W}}$  that exists in all isotropic, homogeneous deforming solid and fluent continua. Thus, this theory intrinsically has little to do with length scale. Linear constitutive theory for the Cauchy moment tensor is proportional to the symmetric part of the internal rotation gradient tensor. Thus, there is an additional material coefficient in this constitutive theory. Whether this can be used to describe smaller scale physics is not clear. If we recall that  ${}^d\mathbf{J}$  and  $\bar{\mathbf{W}}$  are due to  ${}^d\mathbf{J}$  and  $\bar{\mathbf{L}}$  that are macro measure in classical continuum mechanics; then their use in small scale physics than macro scale is obviously of concern.

## 8. Nonlocal theories

Nonlocal theories introduced by Eringen [103–106] consider the stress at a material point to be influenced by the remote material points that are not its immediate neighbors. The new stresses are defined using classical Cauchy stress tensor and a distance kernel through an integral over the spatial domain of influence. In our view this physics is not obvious in solid continua in which the material points remain connected and the disturbance at material point is a consequence of the immediate neighboring material point transmitting energy to it.

We recall that in the polymers (both solids and fluids) the concept of rheology or memory is well known and is due to stressed long chain molecules returning to stress free state after cessation of external disturbance. This behavior is called stress relaxation defined by integral of memory modulus over  $-\infty$  to present value of time. Thus, in this case the time history influences stress state at a point. In this concept if the time integrals are replaced by spatial domain and the integrand by the Cauchy stress and distance kernel, we have the nonlocal stress concept. This is rather obvious, that the rheology concept at a point, that contains integral over time, is completely non analogous to concept of stress at a material point related to other material points through an integral over the spatial domain.

In short, the work presented here is not related and also does not consider nonlocal stress concept leading to nonlocal theories.

## 9. Applications of the non-classical continuum theories presented in this paper

The applications of the non-classical continuum theories presented in this paper are obviously those in which the physics due to internal rotations and Cosserat rotations (and their rates) play an important role in deriving the mathematical description of the deforming continua. We recall that the internal rotations and the internal rotation rates naturally exist in all deforming isotropic and homogeneous solid continua and fluent continua, thus, the non-classical continuum theories based on internal rotations and internal rotation rates in reality correct deficiencies in the classical continuum theories used currently for solid and fluent continua. As discussed in [section 7](#), our view regarding the use of non-classical continuum theory based on internal rotations and internal rotation rates to describe small scale phenomena using length scale is that since the rotations and rotation rates are due to  ${}^d_a\mathbf{J}$  and  $\bar{\mathbf{W}}$  that are macro measures, it is not immediately obvious how these can be used for scales smaller than macro scale.

The theories presented here are extremely helpful in explaining many inconsistencies in beam, plate and shell theories in providing a rigorous and more comprehensive framework for better and consistent formulations. This work is currently in progress. The first publication on beam theories, their consistency and new formulations is in the process of submission for publication [107].

In our view the real applications of these theories are for nonhomogeneous and nonisotropic media in which inhomogeneity and nonisotropy is due to embedded fibers or inclusions. In such cases the deformation physics is difficult to describe accurately using classical theories obviously require extension of the theories presented here (for homogeneous and isotropic matter) to nonhomogeneous and nonisotropic media using integral form of the conservation and the balance laws and consideration of the physics of deformation of the embedded fibers or inclusions and their interaction with the matrix. In this approach only the fiber or inclusion sizes much larger than the mean free path of the molecules of the matrix can be considered as this would permit use of classical continuum mechanics for fiber or inclusions deformation physics.

Consistency and the generality of the conservation and the balance laws in the theories presented here provide simple means of extending current application (both solid and fluent continua) based on classical continuum mechanics to non-classical continuum theories with opportunity of incorporating more comprehensive physics.

## 10. Summary and conclusions

In this paper non-classical continuum theories have been presented for solid and fluent continua. The two non-classical continuum theories presented in the paper for solid continua consider: (i) internal rotations due to Jacobian of deformation and (ii) internal rotations as well as Cosserat rotations as additional three unknown degrees of freedom at a material point. Both internal and Cosserat rotations act about the axes of the same triad located at each material point, the axes of the triad being parallel to the fixed

$x$ -frame. Similar non-classical continuum theories have also been presented for fluent continua in which internal rotation rates due to velocity gradient tensor  $\bar{\mathbf{L}}$  and internal rotation rates as well as Cosserat rotation rates are considered.

It is shown that the non-classical continuum theories for both solid and fluent continua require additional balance law, *balance of moment of moments* to ensure dynamic equilibrium of the deforming matter in the presence of rotations and rotation rates and the conjugate Cauchy moment tensor. The rate derivation of this balance law has been presented for solid as well as fluent continua. In case of solid continua constitutive theories have been presented for thermoelastic solid and for thermoviscoelastic solids with and without memory. For fluent continua the constitutive theories have been considered for thermoviscous fluids and thermoviscoelastic fluids.

General remarks regarding the features and consistency of these non-classical continuum theories have been elaborated in [section 5](#). Some comparison of the features of these theories with Eringen's work on micropolar theories is presented in [section 6](#). Parallelism and contrast in terms of fundamental consideration of the internal rotation and rotation rate non-classical continuum theories for solids and fluids presented here with the published works on couple stress theories is presented in [section 7](#). Nonlocal theories are briefly mentioned in [section 8](#). Elaborate discussion on these theories is not considered in this section as these theories are not related to the present work. Some applications of the present non-classical continuum theories are discussed in [section 9](#). The significant aspects of the present work discussed in [section 5 – 9](#) are not repeated here for the sake of brevity.

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## Disclosure statement

No potential conflict of interest was reported by the authors.

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